A New Monotonic, Clone-Independent, Reversal Symmetric, and Condorcet-Consistent Single-Winner Election Method

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Summary. In recent years, the Pirate Party of Sweden, the Wikimedia Foundation, the Debian project, the "Software in the Public Interest" project, the Gentoo project, and many other private organizations adopted a new single-winner election method for internal elections and referendums. In this paper, we will introduce this method, demonstrate that it satisfies e.g. resolvability, Condorcet, Schwartz, Smith-IIA, Pareto, reversal symmetry, monotonicity, prudence, and independence of clones and present an O(C^3) algorithm to calculate the winner, where C is the number of alternatives.

Keywords and Phrases: Condorcet criterion, independence of clones, monotonicity, Pareto efficiency, reversal symmetry, single-winner election methods, prudent ranking rules

JEL Classification Number: D71

Contents

Symbols

$\wedge$	... and ...
$\vee$	... or ...
$\forall$	... for all ...
$\exists$	... there is at least one ...
$\in$	... element of ...
$\notin$	... not element of ...
$\mathbb{N}$	natural numbers without zero, $\mathbb{N} = \{1, 2, 3, ...\}$
$\mathbb{N}_0$	natural numbers with zero, $\mathbb{N}_0 = \{0, 1, 2, 3, ...\}$
$\mathbb{R}$	real numbers

1. Introduction

One important property of a good single-winner election method is that it minimizes the number of “overruled” voters (according to some heuristic). Because of this reason, the Simpson-Kramer method, that always chooses that alternative whose worst pairwise defeat is the weakest, was very popular over a long time. However, in recent years, the Simpson-Kramer method has been criticized by many social choice theorists. Smith (1973) criticizes that this method doesn’t choose from the top-set of alternatives. Tideman (1987) complains that this method is vulnerable to the strategic nomination of a large number of similar alternatives, so-called *clones*. And Saari (1994) rejects this method for violating *reversal symmetry*. A violation of reversal symmetry can lead to strange situations where still the same alternative is chosen when all ballots are reversed, meaning that the same alternative is identified as best one and simultaneously as worst one.

In this paper, we will show that only a slight modification (section 4.8) of the Simpson-Kramer method is needed so that the resulting method satisfies the criteria proposed by Smith (section 4.7), Tideman (section 4.6), and Saari (section 4.4). The resulting method will be called *Schulze method*. Random simulations by Wright (2009) confirmed that, in almost 99% of all instances, the Schulze method conforms with the Simpson-Kramer method (table 9.1). In this paper, we will prove that, nevertheless, the Schulze method still satisfies all important criteria that are also satisfied by the Simpson-Kramer method, like resolvability (section 4.2), Pareto (section 4.3), monotonicity (section 4.5), and prudence (section 4.9). Because of these reasons, already several private organizations have adopted the Schulze method.

1997 – 2006: In 1997, I proposed the Schulze method to a large number of people, who are interested in mathematical aspects of election methods. In January 2003, the “Software in the Public Interest” (SPI) project, a software developer organization with about 300 eligible members, adopted this method. In June 2003, the Debian project, a software developer organization with about 1,000 eligible members, adopted this method in a referendum with 144 against 16 votes; Debian GNU/Linux is the largest and most popular non-commercial Linux distribution. In May 2005, the Gentoo Foundation, a software developer organization with about 200 eligible members, adopted this method; Gentoo Linux is another wide-spread Linux distribution.

2007 – 2011: In 2008, 2009, and 2011, the Wikimedia Foundation, a non-profit charitable organization with about 43,000 eligible members (in 2011), used the proposed method for the election of its Board of

Trustees; the Wikimedia Foundation is the umbrella organization e.g. for Wikipedia, Wiktionary, Wikiquote, Wikidata, Wikibooks, Wikisource, Wikinews, Wikivoyage, Wikiversity, and Wikispecies; it is, therefore, the fifth most important Internet corporation (after Alphabet/Google/YouTube, Facebook/WhatsApp, Yahoo!, and Baidu). In June 2008, the "Free Software Foundation Europe" (FSFE), a software project with about 1,500 eligible members, adopted this method. In July 2008, Ubuntu, a software developer organization with about 700 eligible members, adopted this method. In October 2009, the "Pirate Party of Sweden" (about 4,000 eligible members) adopted this method. In May 2010, the "Pirate Party of Germany" (about 12,000 eligible members) adopted this method. In November 2010, OpenStack, a software project with about 3,000 eligible members, adopted this method. Since February 2011, the "Pirate Party of Austria" (about 300 eligible members) uses this method. Since November 2011, the "Pirate Party of Australia" (about 1,300 eligible members) uses this method.

2012 – 2017: Since January 2013, the "Pirate Party of Iceland" (about 4,000 eligible members) uses this method. Since April 2013, the associated student government at Northwestern University (about 20,000 eligible members) uses this method. Since October 2013, the "German Association of Pediatricians" ("Berufsverband der Kinder- und Jugendärzte"; BVKJ; about 12,000 eligible members) uses this method. Since October 2013, the "Five Star Movement" ("Movimento 5 Stelle", M5S), a political party in Italy with about 140,000 eligible members, uses this method. Since May 2014, the associated student government at Albert Ludwig University of Freiburg (about 25,000 eligible members) uses this method. In February 2016, the city of Silla (about 19,000 inhabitants) in Spain adopted the Schulze method for referendums (www01 – www05). In July 2016, the "European Students' Forum" ("Association des états généraux des étudiants de l'Europe", AEGEE), a student organization with about 13,000 eligible members, adopted this method. Since January 2017, Podemos, a political party in Spain with about 460,000 eligible members, uses this method.

Today (March 2017), the proposed method is used by more than 60 organizations with more than 700,000 eligible members in total. Therefore, the proposed method is more wide-spread than all other Condorcet-consistent single-winner election methods combined.

Furthermore, the proposed method is used by many Internet decision support systems, like the "Condorcet Internet Voting Service" (CIVS), GoogleVotes (Hardt and Lopes, 2015), LiquidFeedback (Behrens, 2014), Selectricity (Hill, 2008), Airesis, preftools, OpenAgora, and OpenSTV.

There has been some debate about an appropriate name for this method. Some people suggested names like "beatpath", "beatpath method", "beatpath winner", "beatpath power ranking" (BeatPower), "path method", "path voting", "path winner", "Schwartz sequential dropping" (SSD), and "cloneproof Schwartz sequential dropping" (CSSD). Brearley (1999) suggested names like "descending minimum gross score" (DminGS), "descending minimum augmented gross score" (DminAGS), and "descending minimum doubly augmented gross score" (DminDAGS), depending on how the strength of a pairwise link is measured. Heitzig (2001) suggested names like “strong immunity from binary arguments” (SImA) and “sequential dropping towards a spanning tree” (SDST). However, I prefer the name “Schulze method”, not because of academic arrogance, but because the other names do not refer to the method itself but to specific heuristics for implementing it, and so may mislead readers into believing that no other method for implementing it is possible.

In section 2 of this paper, the Schulze method is defined. In section 3, this method is applied to concrete examples. In section 4, this method is analyzed. Detailed descriptions of this method can also be found in publications by Schulze (2003, 2011), Tideman (2006, pages 228–232), Stahl and Johnson (2006, pages 119–130), McCaffrey (2008a, 2008b), Börgers (2009, pages 37–42), Camps (2012a, 2012b, 2013, 2014a, 2014b, 2014c), Behrens (2014), and D. Müller (2014, 2015). This method is also described and discussed in papers by Meskanen and Nurmi (2006a, 2006b, 2008), Yue (2007), Nebel (2009), Wright (2009), Rivest and Shen (2010), Gaspers (2012), Grünheid (2012, 2015), Negriu (2012), Parkes and Xia (2012), Happes (2013), Menton (2013a, 2013b), J. Müller (2013), Felsenthal and Tideman (2014), Li (2014), Mattei (2014), Reisch (2014), Schend (2015), Baumeister and Rothe (2016), Bubboloni and Gori (2016), Caragiannis (2016), Diethelm (2016), Fischer (2016), Hemaspaandra (2016), Parkes and Seuken (2016), and Ruiz-Padillo (2016). Applications of the Schulze method are described in papers by Callison-Burch (2009), Arguello (2011a, 2011b), Audhkhasi (2011), Gelder (2011), Maheswari (2012), Muldoon (2012), Oryńczak (2012), Prati (2012), Bohne (2013, 2015), Zhou (2013, 2014), Akbib (2014), Garg (2014), Lawonn (2014), Pallett (2014), Wang (2014), Baer (2015), Bountris (2015), Degeest (2015), Evita (2015), Nguyen (2015), Plösch (2015), Aswatha (2016), Cai (2016), Chen (2016), Mangeli (2016), Rijnsburger (2016), Verdiesen (2016), Xexéo (2016), and Moal (2017).

2. Definition of the Schulze Method

2.1. Preliminaries

We presume that A is a finite and non-empty set of alternatives. $C \in \mathbb{N}$ with $1 < C < \infty$ is the number of alternatives in A.

A binary relation $\succ$ on A is *asymmetric* if it has the following property:

$\forall\ a,b \in A$, exactly one of the following three statements is valid:

1. $a \succ b$.
2. $b \succ a$.
3. $a \approx b$ (where “$a \approx b$” means “neither $a \succ b$ nor $b \succ a$”).

A binary relation $\succ$ on A is *irreflexive* if it has the following property:

$\forall\ a \in A$: $a \approx a$.

A binary relation $\succ$ on A is *transitive* if it has the following property:

$\forall\ a,b,c \in A$: (($a \succ b$ and $b \succ c$) $\Rightarrow$ $a \succ c$).

A binary relation $\succ$ on A is *negatively transitive* if it has the following property (where "$a \succsim b$" means "not $b \succ a$"):

$$\forall\, a,b,c \in A\text{: } (\ (\ a \succsim b \text{ and } b \succsim c\ ) \Rightarrow a \succsim c\ ).$$

A binary relation $\succ$ on A is *linear* (or *total* or *complete*) if it has the following property:

$$\forall\, a,b \in A\text{: } (\ b \in A \setminus \{a\} \Rightarrow (\ a \succ b \text{ or } b \succ a\ )\ ).$$

A *strict partial order* is an asymmetric, irreflexive, and transitive relation. A *strict weak order* is a strict partial order that is also negatively transitive. A *linear order* (or *total order* or *complete order*) is a strict weak order that is also linear. A *profile* is a finite and non-empty list of strict weak orders each on A.

Input of the proposed method is a profile V. $N \in \mathbb{N}$ with $0 < N < \infty$ is the number of strict weak orders in $V := \{ \succ_1, \dots, \succ_N \}$. These strict weak orders will sometimes be called "voters" or "ballots".

Suppose $V_1 := \{ \succ_1, \dots, \succ_{N_1} \}$ and $V_2 := \{ \succ_{1'}, \dots, \succ_{N_2'} \}$ are two profiles each on the same set of alternatives A. Then the concatenation of these two profiles will be denoted $V_1 + V_2 := \{ \succ_1, \dots, \succ_{N_1}, \succ_{1'}, \dots, \succ_{N_2'} \}$.

"$a \succ_v b$" means "voter $v \in V$ strictly prefers alternative $a \in A$ to alternative b". "$a \approx_v b$" means "voter $v \in V$ is indifferent between alternative a and alternative b". "$a \succsim_v b$" means "$a \succ_v b$ or $a \approx_v b$".

Output of the proposed method is (1) a strict partial order $\mathcal{O}$ on A and (2) a set $\emptyset \neq \mathcal{S} \subseteq A$ of potential winners.

A possible implementation of the Schulze method looks as follows:

> Each voter gets a complete list of all alternatives and ranks these alternatives in order of preference. The individual voter may give the same preference to more than one alternative and he may keep alternatives unranked. When a given voter does not rank all alternatives, then this means (1) that this voter strictly prefers all ranked alternatives to all not ranked alternatives and (2) that this voter is indifferent between all not ranked alternatives. The individual voter may also skip preferences; however, skipping preferences has no impact on the result of the elections since only the cast order of the preferences matters, not the absolute numbers.

Suppose $N[e,f] := \| \{ v \in V \mid e \succ_v f \} \|$ is the number of voters who strictly prefer alternative e to alternative f. We presume that the strength of the link ef depends only on $N[e,f]$ and $N[f,e]$. Therefore, the strength of the link ef can be denoted $(N[e,f],N[f,e])$. We presume that a binary relation $\succ_D$ on $\mathbb{N}_0 \times \mathbb{N}_0$ is defined such that the link ef is stronger than the link gh if and only if $(N[e,f],N[f,e]) \succ_D (N[g,h],N[h,g])$. $N[e,f]$ is the *support* for the link ef; $N[f,e]$ is its *opposition*.

Example 1 (*margin*):

When the strength of the link ef is measured by *margin*, then its strength is the difference $N[e,f] - N[f,e]$ between its support $N[e,f]$ and its opposition $N[f,e]$.

$(N[e,f],N[f,e]) \succ_{margin} (N[g,h],N[h,g])$ if and only if $N[e,f] - N[f,e] > N[g,h] - N[h,g]$.

Example 2 (*ratio*):

When the strength of the link ef is measured by *ratio*, then its strength is the ratio $N[e,f] / N[f,e]$ between its support $N[e,f]$ and its opposition $N[f,e]$.

$(N[e,f],N[f,e]) \succ_{ratio} (N[g,h],N[h,g])$ if and only if at least one of the following conditions is satisfied:

1. $N[e,f] > N[f,e]$ and $N[g,h] \leq N[h,g]$.
2. $N[e,f] \geq N[f,e]$ and $N[g,h] < N[h,g]$.
3. $N[e,f] \cdot N[h,g] > N[f,e] \cdot N[g,h]$.
4. $N[e,f] > N[g,h]$ and $N[f,e] \leq N[h,g]$.
5. $N[e,f] \geq N[g,h]$ and $N[f,e] < N[h,g]$.

Example 3 (*winning votes*):

When the strength of the link ef is measured by *winning votes*, then its strength is measured primarily by its support $N[e,f]$.

$(N[e,f],N[f,e]) \succ_{win} (N[g,h],N[h,g])$ if and only if at least one of the following conditions is satisfied:

1. $N[e,f] > N[f,e]$ and $N[g,h] \leq N[h,g]$.
2. $N[e,f] \geq N[f,e]$ and $N[g,h] < N[h,g]$.
3. $N[e,f] > N[f,e]$ and $N[g,h] > N[h,g]$ and $N[e,f] > N[g,h]$.
4. $N[e,f] > N[f,e]$ and $N[g,h] > N[h,g]$ and $N[e,f] = N[g,h]$ and $N[f,e] < N[h,g]$.
5. $N[e,f] < N[f,e]$ and $N[g,h] < N[h,g]$ and $N[f,e] < N[h,g]$.
6. $N[e,f] < N[f,e]$ and $N[g,h] < N[h,g]$ and $N[f,e] = N[h,g]$ and $N[e,f] > N[g,h]$.

Example 4 (*losing votes*):

When the strength of the link ef is measured by *losing votes*, then its strength is measured primarily by its opposition $N[f,e]$.

$(N[e,f],N[f,e]) \succ_{los} (N[g,h],N[h,g])$ if and only if at least one of the following conditions is satisfied:

1. $N[e,f] > N[f,e]$ and $N[g,h] \leq N[h,g]$.
2. $N[e,f] \geq N[f,e]$ and $N[g,h] < N[h,g]$.
3. $N[e,f] > N[f,e]$ and $N[g,h] > N[h,g]$ and $N[f,e] < N[h,g]$.
4. $N[e,f] > N[f,e]$ and $N[g,h] > N[h,g]$ and $N[f,e] = N[h,g]$ and $N[e,f] > N[g,h]$.
5. $N[e,f] < N[f,e]$ and $N[g,h] < N[h,g]$ and $N[e,f] > N[g,h]$.
6. $N[e,f] < N[f,e]$ and $N[g,h] < N[h,g]$ and $N[e,f] = N[g,h]$ and $N[f,e] < N[h,g]$.

The most intuitive definitions for the strength of a link are its *margin* and its *ratio*. However, we only presume that $\succ_D$ is a strict weak order on $\mathbb{N}_0 \times \mathbb{N}_0$.

For some proofs, we have to make additional presumptions for $\succ_D$. We will state explicitly when and where we take use of additional presumptions. Typical additional presumptions for $\succ_D$ are:

(2.1.1) (*positive responsiveness*)

$$\forall\ (x_1,x_2),(y_1,y_2) \in \mathbb{N}_0 \times \mathbb{N}_0:$$
$$(\ (\ x_1 > y_1 \wedge x_2 \leq y_2\ ) \vee (\ x_1 \geq y_1 \wedge x_2 < y_2\ )\ ) \Rightarrow (x_1,x_2) \succ_D (y_1,y_2).$$

(2.1.2) (*reversal symmetry*)

$$\forall\ (x_1,x_2),(y_1,y_2) \in \mathbb{N}_0 \times \mathbb{N}_0:$$
$$(x_1,x_2) \succ_D (y_1,y_2) \Rightarrow (y_2,y_1) \succ_D (x_2,x_1).$$

(2.1.3) (*homogeneity*)

$$\forall\ (x_1,x_2),(y_1,y_2) \in \mathbb{N}_0 \times \mathbb{N}_0\ \forall\ c_1,c_2 \in \mathbb{N}:$$
$$(c_1{\cdot}x_1,c_1{\cdot}x_2) \succ_D (c_1{\cdot}y_1,c_1{\cdot}y_2) \Rightarrow (c_2{\cdot}x_1,c_2{\cdot}x_2) \succ_D (c_2{\cdot}y_1,c_2{\cdot}y_2).$$

The presumption, that the strength of the link *ef* depends only on $N[e,f]$ and $N[f,e]$, guarantees (1) that the proposed method satisfies anonymity and neutrality, (2) that adding a ballot, on which all alternatives are ranked equally, cannot change the result of the elections, and (3) that the proposed method is a C2 *Condorcet social choice function* (CSCF) according to Fishburn's (1977) terminology.

Presumption (2.1.1) says that, when the support of a link increases and its opposition doesn't increase or when its opposition decreases and its support doesn't decrease, then the strength of this link increases. So presumption (2.1.1) says that the strength of a link responses to a change of its support or its opposition in the correct manner. Presumption (2.1.1) guarantees that the proposed method satisfies resolvability (section 4.2), Pareto (section 4.3), and monotonicity (section 4.5). When each voter $v \in V$ casts a linear order $\succ_v$ on A, then all definitions for $\succ_D$, that satisfy presumption (2.1.1), are identical.

Presumption (2.1.2) says that, the stronger the link (x_1,x_2) gets, the weaker the opposite link (x_2,x_1) gets. Presumption (2.1.2) basically says that, when the individual ballots $\succ_v$ are reversed for all voters $v \in V$, then also the order of the links $(x_1,x_2) \succ_D (y_1,y_2)$ is reversed.

Homogeneity means that the result depends only on the proportion of ballots of each type, not on their absolute numbers. Presumption (2.1.3) guarantees that the proposed method satisfies homogeneity.

$\succ_{margin}$, $\succ_{ratio}$, $\succ_{win}$, and $\succ_{los}$ each satisfy (2.1.1) – (2.1.3).

Corollary (2.1.4):

If $\succ_D$ satisfies presumption (2.1.2), then all ties have equivalent strengths. In short:

(2.1.4) $\forall\, x,y \in \mathbb{N}_0$: $(x,x) \approx_D (y,y)$.

Proof of corollary (2.1.4):

Suppose $(x,x) \succ_D (y,y)$ for some $x,y \in \mathbb{N}_0$. Then with (2.1.2), we get $(y,y) \succ_D (x,x)$. But this is a contradiction to the presumption $(x,x) \succ_D (y,y)$ and to the presumption that $\succ_D$ is a strict weak order. □

Corollary (2.1.5):

If $\succ_D$ satisfies presumptions (2.1.1) and (2.1.2), then (i) every pairwise victory is stronger than every pairwise tie and (ii) every pairwise tie is stronger than every pairwise defeat. In short:

(2.1.5) (*majority*)

$\forall\, (x_1,x_2),(y_1,y_2) \in \mathbb{N}_0 \times \mathbb{N}_0$:
$(\,(\,x_1 > x_2 \wedge y_1 \leq y_2\,) \vee (\,x_1 \geq x_2 \wedge y_1 < y_2\,)\,) \Rightarrow (x_1,x_2) \succ_D (y_1,y_2)$.

Proof of corollary (2.1.5):

Suppose $(x_1,x_2) \in \mathbb{N}_0 \times \mathbb{N}_0$ with $x_1 > x_2$ is a victory.

Suppose $(y_1,y_2) \in \mathbb{N}_0 \times \mathbb{N}_0$ with $y_1 = y_2$ is a tie.

Suppose $(z_1,z_2) \in \mathbb{N}_0 \times \mathbb{N}_0$ with $z_1 < z_2$ is a defeat.

With (2.1.1), we get: $(x_1,x_2) \succ_D (x_2,x_2)$.

With (2.1.4), we get: $(x_2,x_2) \approx_D (y_1,y_2)$.

With (2.1.4), we get: $(y_1,y_2) \approx_D (z_1,z_1)$.

With (2.1.1), we get: $(z_1,z_1) \succ_D (z_1,z_2)$.

Therefore, we get: $(x_1,x_2) \succ_D (x_2,x_2) \approx_D (y_1,y_2) \approx_D (z_1,z_1) \succ_D (z_1,z_2)$.

Thus, we get (2.1.5). □

Suppose $\emptyset \neq \mathcal{M} \subset \mathbb{N}_0 \times \mathbb{N}_0$ is finite and non-empty. Then “$\max_D\mathcal{M}$”, the *set of maximum elements* of $\mathcal{M}$, and “$\min_D\mathcal{M}$”, the *set of minimum elements* of $\mathcal{M}$, are defined as follows: $(\beta_1,\beta_2) \in \max_D\mathcal{M}$ if and only if (1) $(\beta_1,\beta_2) \in \mathcal{M}$ and (2) $(\beta_1,\beta_2) \succsim_D (\delta_1,\delta_2)\ \forall\ (\delta_1,\delta_2) \in \mathcal{M}$. $(\gamma_1,\gamma_2) \in \min_D\mathcal{M}$ if and only if (1) $(\gamma_1,\gamma_2) \in \mathcal{M}$ and (2) $(\gamma_1,\gamma_2) \precsim_D (\delta_1,\delta_2)\ \forall\ (\delta_1,\delta_2) \in \mathcal{M}$.

We write “$(\beta_1,\beta_2) := \max_D\mathcal{M}$” and “$(\gamma_1,\gamma_2) := \min_D\mathcal{M}$” for “$(\beta_1,\beta_2)$ is an arbitrarily chosen element of $\max_D\mathcal{M}$” and “(γ_1,γ_2) is an arbitrarily chosen element of $\min_D\mathcal{M}$”.

2.2. Basic Definitions

In this section, the Schulze method is defined. Concrete examples can be found in section 3.

Basic idea of the Schulze method is that the *strength* of the indirect comparison “alternative a vs. alternative b” is the *strength* of the *strongest path* $a \equiv c(1),...,c(n) \equiv b$ from alternative $a \in A$ to alternative $b \in A \setminus \{a\}$ and that the *strength* of a path is the *strength* $(N[c(i),c(i+1)],N[c(i+1),c(i)])$ of its *weakest link* $c(i),c(i+1)$.

The Schulze method is defined as follows:

> A *path* from alternative $x \in A$ to alternative $y \in A \setminus \{x\}$ is a sequence of alternatives $c(1),...,c(n) \in A$ with the following properties:
>
> 1. $x \equiv c(1)$.
> 2. $y \equiv c(n)$.
> 3. $n \in \mathbb{N}$ with $2 \leq n < \infty$.
> 4. For all $i = 1,...,(n-1)$: $c(i+1) \in A \setminus \{c(i)\}$.
>
> The *strength* of the path $c(1),...,c(n)$ is
> $$\min_D \{ (N[c(i),c(i+1)],N[c(i+1),c(i)]) \mid i = 1,...,(n-1) \}.$$
>
> In other words: The strength of a path is the strength of its weakest link.
>
> When a path $c(1),...,c(n)$ has the strength $(z_1,z_2) \in \mathbb{N}_0 \times \mathbb{N}_0$, then the *critical links* of this path are the links with $(N[c(i),c(i+1)],N[c(i+1),c(i)]) \approx_D (z_1,z_2)$.
>
> $$P_D[a,b] := \max_D \{ \min_D \{ (N[c(i),c(i+1)],N[c(i+1),c(i)]) \mid i = 1,...,(n-1) \} \mid c(1),...,c(n) \text{ is a path from alternative } a \text{ to alternative } b \}.$$
>
> In other words: $P_D[a,b] \in \mathbb{N}_0 \times \mathbb{N}_0$ is the strength of the strongest path from alternative $a \in A$ to alternative $b \in A \setminus \{a\}$.
>
> (2.2.1) The binary relation $\mathcal{O}$ on A is defined as follows:
> $$ab \in \mathcal{O} :\Leftrightarrow P_D[a,b] \succ_D P_D[b,a].$$
>
> (2.2.2) $\mathcal{S} := \{ a \in A \mid \forall\, b \in A \setminus \{a\}: ba \notin \mathcal{O} \}$ is the *set of potential winners*.

When there is only one potential winner $\mathcal{S} = \{a\}$, then this alternative is a *unique winner*.

When $P_D[a,b] \succ_D P_D[b,a]$, then we say “alternative a disqualifies alternative b” or “alternative a dominates alternative b”.

As the link ab is already a path from alternative a to alternative b of strength $(N[a,b],N[b,a])$, we get

(2.2.3) $\quad \forall\ a,b \in A: P_D[a,b] \succsim_D (N[a,b],N[b,a])$.

With (2.2.1) and (2.2.3), we get

(2.2.4) $\quad (N[a,b],N[b,a]) \succ_D P_D[b,a] \Rightarrow ab \in \mathcal{O}$.

Furthermore, we get

(2.2.5) $\quad \forall\ a,b,c \in A: \min_D \{\ P_D[a,b], P_D[b,c]\ \} \precsim_D P_D[a,c]$.

Otherwise, if $\min_D \{\ P_D[a,b], P_D[b,c]\ \}$ was strictly larger than $P_D[a,c]$, then this would be a contradiction to the definition of $P_D[a,c]$ since there would be a path from alternative a to alternative c via alternative b with a strength of more than $P_D[a,c]$.

Furthermore, we get

(2.2.6) $\quad \forall\ a,b \in A: P_D[a,b] \precsim_D \max_D \{\ (N[a,c],N[c,a]) \mid c \in A \setminus \{a\}\ \}$.

(2.2.7) $\quad \forall\ a,b \in A: P_D[a,b] \precsim_D \max_D \{\ (N[c,b],N[b,c]) \mid c \in A \setminus \{b\}\ \}$.

The asymmetry of $\mathcal{O}$ follows directly from the asymmetry of $\succ_D$. The irreflexivity of $\mathcal{O}$ follows directly from the irreflexivity of $\succ_D$. Furthermore, in section 4.1, we will see that the binary relation $\mathcal{O}$ is transitive. This guarantees that there is always at least one potential winner.

Suppose $\varnothing \neq B \subsetneq A$. Then we get

(2.2.8) $\quad \forall\ a \in B\ \forall\ b \notin B: P_D[a,b] \precsim_D \max_D \{\ (N[c,d],N[d,c]) \mid c \in B \text{ and } d \notin B\ \}$.

2.3. Implementation

The strength $P_D[i,j]$ of the strongest path from alternative $i \in A$ to alternative $j \in A \setminus \{i\}$ can be calculated with the Floyd (1962) algorithm. The runtime to calculate the strengths of all strongest paths is O(C^3), where C is the number of alternatives in A.

Input: $N[i,j] \in \mathbb{N}_0$ is the number of voters who strictly prefer alternative $i \in A$ to alternative $j \in A \setminus \{i\}$.

Output: $P_D[i,j] \in \mathbb{N}_0 \times \mathbb{N}_0$ is the strength of the strongest path from alternative $i \in A$ to alternative $j \in A \setminus \{i\}$.

$pred[i,j] \in A \setminus \{j\}$ is the predecessor of alternative j in the strongest path from alternative $i \in A$ to alternative $j \in A \setminus \{i\}$.

$\mathcal{O}$ is the binary relation as defined in (2.2.1).

“$winner[i] = true$” if and only if $i \in \mathcal{S}$.

Stage 1 (initialization):

```
for i : = 1 to C
begin
    for j : = 1 to C
    begin
        if ( i ≠ j ) then
        begin
            P_D[i,j] : = (N[i,j],N[j,i])
            pred[i,j] : = i
        end
    end
end
```

Stage 2 (calculation of the strengths of the strongest paths):

```
for i : = 1 to C
begin
    for j : = 1 to C
    begin
        if ( i ≠ j ) then
        begin
            for k : = 1 to C
            begin
                if ( i ≠ k ) then
                begin
                    if ( j ≠ k ) then
                    begin
                        if ( P_D[j,k] ≺_D min_D { P_D[j,i], P_D[i,k] } ) then
                        begin
                            P_D[j,k] : = min_D { P_D[j,i], P_D[i,k] }
                            if ( pred[j,k] ≠ pred[i,k] ) then
                            begin
                                pred[j,k] : = pred[i,k]
                            end
                        end
                    end
                end
            end
        end
    end
end
```

Stage 3 (calculation of the binary relation $\mathcal{O}$ and the set of potential winners):

```
for i : = 1 to C
begin
    winner[i] : = true
    for j : = 1 to C
    begin
        if ( i ≠ j ) then
        begin
            if ( P_D[j,i] ≻_D P_D[i,j] ) then
            begin
                ji ∈ O
                winner[i] : = false
            end
            else
            begin
                ji ∉ O
            end
        end
    end
end
```

(α) It cannot be stressed frequently enough that the order of the indices in the triple-loop of the Floyd algorithm is not irrelevant. When i is the index of the outer loop of the triple-loop of the Floyd algorithm, then the clause (line 24) must be “ if ($P_D[j,k] \prec_D \min_D \{ P_D[j,i], P_D[i,k] \}$) ”. Otherwise, it is not guaranteed that a single pass through the triple-loop of the Floyd algorithm is sufficient to find the strongest paths.

(β) With the predecessor matrix $pred[i,j]$, we can recursively determine the strongest paths. Suppose we want to determine the strongest path $c(1),...,c(n)$ from alternative $a \in A$ to alternative $b \in A \setminus \{a\}$. Then we start with

$$n := 1$$

$$d(1) := b$$

We repeat

$$n := n + 1$$

$$d(n) := pred[a,d(n-1)]$$

until we get $d(n) = a$ for some $n \in \mathbb{N}$. The strongest path $c(1),...,c(n)$ from alternative a to alternative b is then given by $d(n),...,d(1)$.

(γ) The runtime to calculate the pairwise matrix is $O(N \cdot (C^2))$. The runtime of the Floyd algorithm, as defined in this section, is $O(C^3)$. Therefore, the total runtime to calculate the binary relation $\mathcal{O}$, as defined in (2.2.1), and the set $\mathcal{S}$, as defined in (2.2.2), is $O(N \cdot (C^2) + C^3)$.

3. Examples

3.1. Example 1

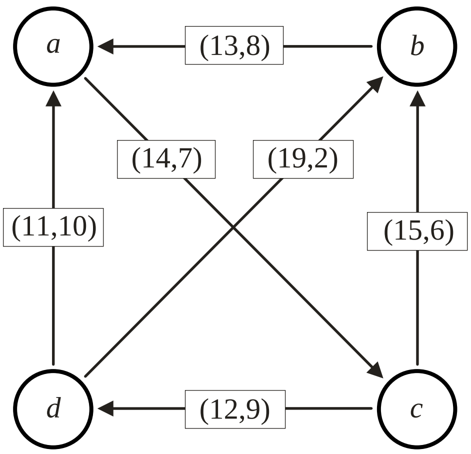
Example 1:

8 voters	$a \succ_v c \succ_v d \succ_v b$
2 voters	$b \succ_v a \succ_v d \succ_v c$
4 voters	$c \succ_v d \succ_v b \succ_v a$
4 voters	$d \succ_v b \succ_v a \succ_v c$
3 voters	$d \succ_v c \succ_v b \succ_v a$

$N[i,j] \in \mathbb{N}_0$ is the number of voters who strictly prefer alternative $i \in A$ to alternative $j \in A \setminus \{i\}$. In example 1, the pairwise matrix N looks as follows:

	$N[*,a]$	$N[*,b]$	$N[*,c]$	$N[*,d]$
$N[a,*]$	---	8	14	10
$N[b,*]$	13	---	6	2
$N[c,*]$	7	15	---	12
$N[d,*]$	11	19	9	---

The following digraph illustrates the graph theoretic interpretation of pairwise elections. If $N[i,j] > N[j,i]$, then there is a link from vertex i to vertex j of strength $(N[i,j],N[j,i])$:



The above digraph can be used to determine the strengths of the strongest paths. In the following, “x, (Z_1,Z_2), y” means “$(N[x,y],N[y,x]) = (Z_1,Z_2)$”.

$a \rightarrow b$: There are 2 paths from alternative a to alternative b.

Path 1: a, (14,7), c, (15,6), b
with a strength of $\min_D$ { (14,7), (15,6) } $\approx_D$ (14,7).

Path 2: a, (14,7), c, (12,9), d, (19,2), b
with a strength of $\min_D$ { (14,7), (12,9), (19,2) } $\approx_D$ (12,9).

So the strength of the strongest path from alternative a to alternative b is $\max_D$ { (14,7), (12,9) } $\approx_D$ (14,7).

$a \rightarrow c$: There is only one path from alternative a to alternative c.

Path 1: a, (14,7), c with a strength of (14,7).

$a \rightarrow d$: There is only one path from alternative a to alternative d.

Path 1: a, (14,7), c, (12,9), d
with a strength of $\min_D$ { (14,7), (12,9) } $\approx_D$ (12,9).

$b \rightarrow a$: There is only one path from alternative b to alternative a.

Path 1: b, (13,8), a with a strength of (13,8).

$b \rightarrow c$: There is only one path from alternative b to alternative c.

Path 1: b, (13,8), a, (14,7), c
with a strength of $\min_D$ { (13,8), (14,7) } $\approx_D$ (13,8).

$b \rightarrow d$: There is only one path from alternative b to alternative d.

Path 1: b, (13,8), a, (14,7), c, (12,9), d
with a strength of $\min_D$ { (13,8), (14,7), (12,9) } $\approx_D$ (12,9).

$c \rightarrow a$: There are 3 paths from alternative c to alternative a.

Path 1: c, (15,6), b, (13,8), a
with a strength of $\min_D$ { (15,6), (13,8) } $\approx_D$ (13,8).

Path 2: c, (12,9), d, (11,10), a
with a strength of $\min_D$ { (12,9), (11,10) } $\approx_D$ (11,10).

Path 3: c, (12,9), d, (19,2), b, (13,8), a
with a strength of $\min_D$ { (12,9), (19,2), (13,8) } $\approx_D$ (12,9).

So the strength of the strongest path from alternative c to alternative a is $\max_D$ { (13,8), (11,10), (12,9) } $\approx_D$ (13,8).

$c \rightarrow b$: There are 2 paths from alternative c to alternative b.

Path 1: c, (15,6), b with a strength of (15,6).

Path 2: c, (12,9), d, (19,2), b
with a strength of $\min_D$ { (12,9), (19,2) } $\approx_D$ (12,9).

So the strength of the strongest path from alternative c to alternative b is $\max_D$ { (15,6), (12,9) } $\approx_D$ (15,6).

$c \rightarrow d$: There is only one path from alternative c to alternative d.

Path 1: c, (12,9), d with a strength of (12,9).

$d \rightarrow a$: There are 2 paths from alternative d to alternative a.

Path 1: d, (11,10), a with a strength of (11,10).

Path 2: d, (19,2), b, (13,8), a
with a strength of $\min_D$ { (19,2), (13,8) } $\approx_D$ (13,8).

So the strength of the strongest path from alternative d to alternative a is $\max_D$ { (11,10), (13,8) } $\approx_D$ (13,8).

$d \rightarrow b$: There are 2 paths from alternative d to alternative b.

Path 1: d, (11,10), a, (14,7), c, (15,6), b
with a strength of $\min_D$ { (11,10), (14,7), (15,6) } $\approx_D$ (11,10).

Path 2: d, (19,2), b with a strength of (19,2).

So the strength of the strongest path from alternative d to alternative b is $\max_D$ { (11,10), (19,2) } $\approx_D$ (19,2).

$d \rightarrow c$: There are 2 paths from alternative d to alternative c.

Path 1: d, (11,10), a, (14,7), c
with a strength of $\min_D$ { (11,10), (14,7) } $\approx_D$ (11,10).

Path 2: d, (19,2), b, (13,8), a, (14,7), c
with a strength of $\min_D$ { (19,2), (13,8), (14,7) } $\approx_D$ (13,8).

So the strength of the strongest path from alternative d to alternative c is $\max_D$ { (11,10), (13,8) } $\approx_D$ (13,8).

The following table lists the strongest paths. The critical links of the strongest paths are underlined:

	... to a	... to b	... to c	... to d
from a ...	---	a, $\underline{(14,7)}$, c, (15,6), b	a, $\underline{(14,7)}$, c	a, (14,7), c, $\underline{(12,9)}$, d
from b ...	b, $\underline{(13,8)}$, a	---	b, $\underline{(13,8)}$, a, (14,7), c	b, (13,8), a, (14,7), c, $\underline{(12,9)}$, d
from c ...	c, (15,6), b, $\underline{(13,8)}$, a	c, $\underline{(15,6)}$, b	---	c, $\underline{(12,9)}$, d
from d ...	d, (19,2), b, $\underline{(13,8)}$, a	d, $\underline{(19,2)}$, b	d, (19,2), b, $\underline{(13,8)}$, a, (14,7), c	---

The strengths of the strongest paths are:

	$P_D[*,a]$	$P_D[*,b]$	$P_D[*,c]$	$P_D[*,d]$
$P_D[a,*]$	---	(14,7)	(14,7)	(12,9)
$P_D[b,*]$	(13,8)	---	(13,8)	(12,9)
$P_D[c,*]$	(13,8)	(15,6)	---	(12,9)
$P_D[d,*]$	(13,8)	(19,2)	(13,8)	---

$xy \in \mathcal{O}$ if and only if $P_D[x,y] \succ_D P_D[y,x]$. So in example 1, we get $\mathcal{O} = \{ab, ac, cb, da, db, dc\}$.

$x \in \mathcal{S}$ if and only if $yx \notin \mathcal{O}$ for all $y \in A \setminus \{x\}$. So in example 1, we get $\mathcal{S} = \{d\}$.

Suppose, the strongest paths are calculated with the Floyd algorithm, as defined in section 2.3. Then the following table documents the 24 steps of the Floyd algorithm.

We start with

- $P_D[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(6,15)	(13,8)	(14,7)	b	a	$P_D[b,c]$ is updated from (6,15) to (13,8); $pred[b,c]$ is updated from b to a
2	a	b	d	(2,19)	(13,8)	(10,11)	b	a	$P_D[b,d]$ is updated from (2,19) to (10,11); $pred[b,d]$ is updated from b to a
3	a	c	b	(15,6)	(7,14)	(8,13)	c	a	
4	a	c	d	(12,9)	(7,14)	(10,11)	c	a	
5	a	d	b	(19,2)	(11,10)	(8,13)	d	a	
6	a	d	c	(9,12)	(11,10)	(14,7)	d	a	$P_D[d,c]$ is updated from (9,12) to (11,10); $pred[d,c]$ is updated from d to a
7	b	a	c	(14,7)	(8,13)	(13,8)	a	a	
8	b	a	d	(10,11)	(8,13)	(10,11)	a	a	
9	b	c	a	(7,14)	(15,6)	(13,8)	c	b	$P_D[c,a]$ is updated from (7,14) to (13,8); $pred[c,a]$ is updated from c to b
10	b	c	d	(12,9)	(15,6)	(10,11)	c	a	
11	b	d	a	(11,10)	(19,2)	(13,8)	d	b	$P_D[d,a]$ is updated from (11,10) to (13,8); $pred[d,a]$ is updated from d to b
12	b	d	c	(11,10)	(19,2)	(13,8)	a	a	$P_D[d,c]$ is updated from (11,10) to (13,8)
13	c	a	b	(8,13)	(14,7)	(15,6)	a	c	$P_D[a,b]$ is updated from (8,13) to (14,7); $pred[a,b]$ is updated from a to c
14	c	a	d	(10,11)	(14,7)	(12,9)	a	c	$P_D[a,d]$ is updated from (10,11) to (12,9); $pred[a,d]$ is updated from a to c
15	c	b	a	(13,8)	(13,8)	(13,8)	b	b	
16	c	b	d	(10,11)	(13,8)	(12,9)	a	c	$P_D[b,d]$ is updated from (10,11) to (12,9); $pred[b,d]$ is updated from a to c
17	c	d	a	(13,8)	(13,8)	(13,8)	b	b	
18	c	d	b	(19,2)	(13,8)	(15,6)	d	c	
19	d	a	b	(14,7)	(12,9)	(19,2)	c	d	
20	d	a	c	(14,7)	(12,9)	(13,8)	a	a	
21	d	b	a	(13,8)	(12,9)	(13,8)	b	b	
22	d	b	c	(13,8)	(12,9)	(13,8)	a	a	
23	d	c	a	(13,8)	(12,9)	(13,8)	b	b	
24	d	c	b	(15,6)	(12,9)	(19,2)	c	d	

3.2. Example 2

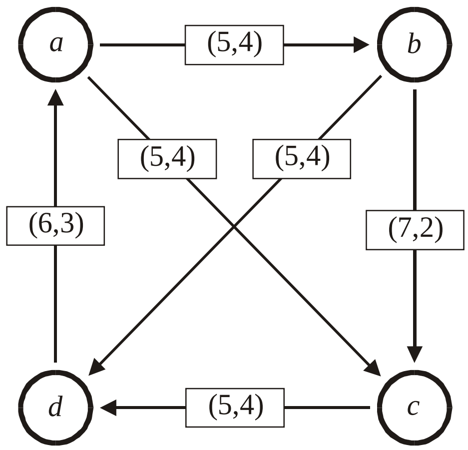
Example 2:

3 voters	$a \succ_v b \succ_v c \succ_v d$
2 voters	$c \succ_v b \succ_v d \succ_v a$
2 voters	$d \succ_v a \succ_v b \succ_v c$
2 voters	$d \succ_v b \succ_v c \succ_v a$

The pairwise matrix N looks as follows:

	$N[*,a]$	$N[*,b]$	$N[*,c]$	$N[*,d]$
$N[a,*]$	---	5	5	3
$N[b,*]$	4	---	7	5
$N[c,*]$	4	2	---	5
$N[d,*]$	6	4	4	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*
from *a* ...	---	*a*, <u>(5,4)</u>, *b*	*a*, <u>(5,4)</u>, *c*	*a*, <u>(5,4)</u>, *b*, <u>(5,4)</u>, *d*
from *b* ...	*b*, <u>(5,4)</u>, *d*, (6,3), *a*	---	*b*, <u>(7,2)</u>, *c*	*b*, <u>(5,4)</u>, *d*
from *c* ...	*c*, <u>(5,4)</u>, *d*, (6,3), *a*	*c*, <u>(5,4)</u>, *d*, (6,3), *a*, <u>(5,4)</u>, *b*	---	*c*, <u>(5,4)</u>, *d*
from *d* ...	*d*, <u>(6,3)</u>, *a*	*d*, (6,3), *a*, <u>(5,4)</u>, *b*	*d*, (6,3), *a*, <u>(5,4)</u>, *c*	---

Therefore, the strengths of the strongest paths are:

	$P_D[*,a]$	$P_D[*,b]$	$P_D[*,c]$	$P_D[*,d]$
$P_D[a,*]$	---	(5,4)	(5,4)	(5,4)
$P_D[b,*]$	(5,4)	---	(7,2)	(5,4)
$P_D[c,*]$	(5,4)	(5,4)	---	(5,4)
$P_D[d,*]$	(6,3)	(5,4)	(5,4)	---

We get $\mathcal{O} = \{bc, da\}$ and $\mathcal{S} = \{b, d\}$.

Suppose, the strongest paths are calculated with the Floyd algorithm, as defined in section 2.3. Then the following table documents the 24 steps of the Floyd algorithm.

We start with

- $P_D[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(7,2)	(4,5)	(5,4)	b	a	
2	a	b	d	(5,4)	(4,5)	(3,6)	b	a	
3	a	c	b	(2,7)	(4,5)	(5,4)	c	a	$P_D[c,b]$ is updated from (2,7) to (4,5); $pred[c,b]$ is updated from c to a
4	a	c	d	(5,4)	(4,5)	(3,6)	c	a	
5	a	d	b	(4,5)	(6,3)	(5,4)	d	a	$P_D[d,b]$ is updated from (4,5) to (5,4); $pred[d,b]$ is updated from d to a
6	a	d	c	(4,5)	(6,3)	(5,4)	d	a	$P_D[d,c]$ is updated from (4,5) to (5,4); $pred[d,c]$ is updated from d to a
7	b	a	c	(5,4)	(5,4)	(7,2)	a	b	
8	b	a	d	(3,6)	(5,4)	(5,4)	a	b	$P_D[a,d]$ is updated from (3,6) to (5,4); $pred[a,d]$ is updated from a to b
9	b	c	a	(4,5)	(4,5)	(4,5)	c	b	
10	b	c	d	(5,4)	(4,5)	(5,4)	c	b	
11	b	d	a	(6,3)	(5,4)	(4,5)	d	b	
12	b	d	c	(5,4)	(5,4)	(7,2)	a	b	
13	c	a	b	(5,4)	(5,4)	(4,5)	a	a	
14	c	a	d	(5,4)	(5,4)	(5,4)	b	c	
15	c	b	a	(4,5)	(7,2)	(4,5)	b	c	
16	c	b	d	(5,4)	(7,2)	(5,4)	b	c	
17	c	d	a	(6,3)	(5,4)	(4,5)	d	c	
18	c	d	b	(5,4)	(5,4)	(4,5)	a	a	
19	d	a	b	(5,4)	(5,4)	(5,4)	a	a	
20	d	a	c	(5,4)	(5,4)	(5,4)	a	a	
21	d	b	a	(4,5)	(5,4)	(6,3)	b	d	$P_D[b,a]$ is updated from (4,5) to (5,4); $pred[b,a]$ is updated from b to d
22	d	b	c	(7,2)	(5,4)	(5,4)	b	a	
23	d	c	a	(4,5)	(5,4)	(6,3)	c	d	$P_D[c,a]$ is updated from (4,5) to (5,4); $pred[c,a]$ is updated from c to d
24	d	c	b	(4,5)	(5,4)	(5,4)	a	a	$P_D[c,b]$ is updated from (4,5) to (5,4)

3.3. Example 3

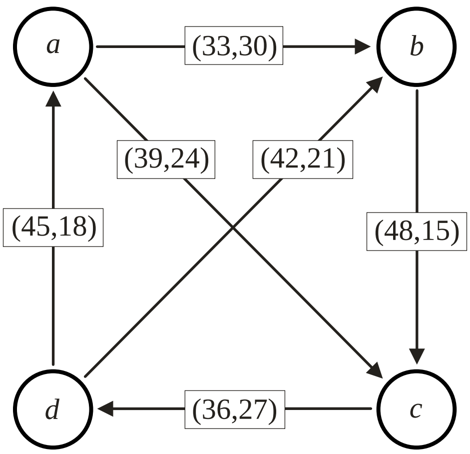
Example 3:

12 voters	$a \succ_v b \succ_v c \succ_v d$
6 voters	$a \succ_v d \succ_v b \succ_v c$
9 voters	$b \succ_v c \succ_v d \succ_v a$
15 voters	$c \succ_v d \succ_v a \succ_v b$
21 voters	$d \succ_v b \succ_v a \succ_v c$

The pairwise matrix *N* looks as follows:

	$N[*,a]$	$N[*,b]$	$N[*,c]$	$N[*,d]$
$N[a,*]$	---	33	39	18
$N[b,*]$	30	---	48	21
$N[c,*]$	24	15	---	36
$N[d,*]$	45	42	27	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*
from *a* ...	---	*a*, (39,24), *c*, (36,27), *d*, (42,21), *b*	*a*, (39,24), *c*	*a*, (39,24), *c*, (36,27), *d*
from *b* ...	*b*, (48,15), *c*, (36,27), *d*, (45,18), *a*	---	*b*, (48,15), *c*	*b*, (48,15), *c*, (36,27), *d*
from *c* ...	*c*, (36,27), *d*, (45,18), *a*	*c*, (36,27), *d*, (42,21), *b*	---	*c*, (36,27), *d*
from *d* ...	*d*, (45,18), *a*	*d*, (42,21), *b*	*d*, (42,21), *b*, (48,15), *c*	---

Therefore, the strengths of the strongest paths are:

	$P_D[*,a]$	$P_D[*,b]$	$P_D[*,c]$	$P_D[*,d]$
$P_D[a,*]$	---	(36,27)	(39,24)	(36,27)
$P_D[b,*]$	(36,27)	---	(48,15)	(36,27)
$P_D[c,*]$	(36,27)	(36,27)	---	(36,27)
$P_D[d,*]$	(45,18)	(42,21)	(42,21)	---

We get $\mathcal{O} = \{ac, bc, da, db, dc\}$ and $\mathcal{S} = \{d\}$.

Suppose, the strongest paths are calculated with the Floyd algorithm, as defined in section 2.3. Then the following table documents the 24 steps of the Floyd algorithm.

We start with

- $P_D[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(48,15)	(30,33)	(39,24)	b	a	
2	a	b	d	(21,42)	(30,33)	(18,45)	b	a	
3	a	c	b	(15,48)	(24,39)	(33,30)	c	a	$P_D[c,b]$ is updated from (15,48) to (24,39); $pred[c,b]$ is updated from c to a
4	a	c	d	(36,27)	(24,39)	(18,45)	c	a	
5	a	d	b	(42,21)	(45,18)	(33,30)	d	a	
6	a	d	c	(27,36)	(45,18)	(39,24)	d	a	$P_D[d,c]$ is updated from (27,36) to (39,24); $pred[d,c]$ is updated from d to a
7	b	a	c	(39,24)	(33,30)	(48,15)	a	b	
8	b	a	d	(18,45)	(33,30)	(21,42)	a	b	$P_D[a,d]$ is updated from (18,45) to (21,42); $pred[a,d]$ is updated from a to b
9	b	c	a	(24,39)	(24,39)	(30,33)	c	b	
10	b	c	d	(36,27)	(24,39)	(21,42)	c	b	
11	b	d	a	(45,18)	(42,21)	(30,33)	d	b	
12	b	d	c	(39,24)	(42,21)	(48,15)	a	b	$P_D[d,c]$ is updated from (39,24) to (42,21); $pred[d,c]$ is updated from a to b
13	c	a	b	(33,30)	(39,24)	(24,39)	a	a	
14	c	a	d	(21,42)	(39,24)	(36,27)	b	c	$P_D[a,d]$ is updated from (21,42) to (36,27); $pred[a,d]$ is updated from b to c
15	c	b	a	(30,33)	(48,15)	(24,39)	b	c	
16	c	b	d	(21,42)	(48,15)	(36,27)	b	c	$P_D[b,d]$ is updated from (21,42) to (36,27); $pred[b,d]$ is updated from b to c
17	c	d	a	(45,18)	(42,21)	(24,39)	d	c	
18	c	d	b	(42,21)	(42,21)	(24,39)	d	a	
19	d	a	b	(33,30)	(36,27)	(42,21)	a	d	$P_D[a,b]$ is updated from (33,30) to (36,27); $pred[a,b]$ is updated from a to d
20	d	a	c	(39,24)	(36,27)	(42,21)	a	b	
21	d	b	a	(30,33)	(36,27)	(45,18)	b	d	$P_D[b,a]$ is updated from (30,33) to (36,27); $pred[b,a]$ is updated from b to d
22	d	b	c	(48,15)	(36,27)	(42,21)	b	b	
23	d	c	a	(24,39)	(36,27)	(45,18)	c	d	$P_D[c,a]$ is updated from (24,39) to (36,27); $pred[c,a]$ is updated from c to d
24	d	c	b	(24,39)	(36,27)	(42,21)	a	d	$P_D[c,b]$ is updated from (24,39) to (36,27); $pred[c,b]$ is updated from a to d

3.4. Example 4

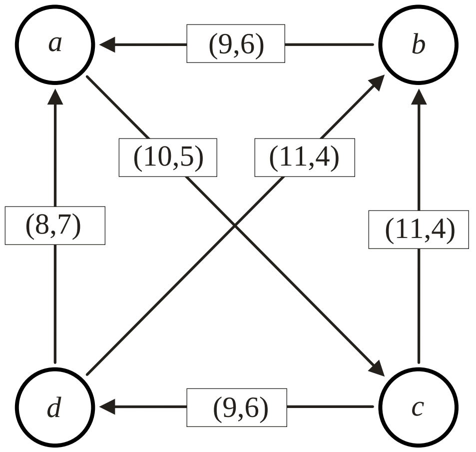
Example 4:

6	voters	$a \succ_v c \succ_v d \succ_v b$
1	voter	$b \succ_v a \succ_v d \succ_v c$
3	voters	$c \succ_v b \succ_v d \succ_v a$
3	voters	$d \succ_v b \succ_v a \succ_v c$
2	voters	$d \succ_v c \succ_v b \succ_v a$

The pairwise matrix N looks as follows:

	$N[*,a]$	$N[*,b]$	$N[*,c]$	$N[*,d]$
$N[a,*]$	---	6	10	7
$N[b,*]$	9	---	4	4
$N[c,*]$	5	11	---	9
$N[d,*]$	8	11	6	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*
from *a* ...	---	*a*, <u>(10,5)</u>, *c*, (11,4), *b*	*a*, <u>(10,5)</u>, *c*	*a*, (10,5), *c*, <u>(9,6)</u>, *d*
from *b* ...	*b*, <u>(9,6)</u>, *a*	---	*b*, <u>(9,6)</u>, *a*, (10,5), *c*	*b*, <u>(9,6)</u>, *a*, (10,5), *c*, <u>(9,6)</u>, *d*
from *c* ...	*c*, (11,4), *b*, <u>(9,6)</u>, *a*	*c*, <u>(11,4)</u>, *b*	---	*c*, <u>(9,6)</u>, *d*
from *d* ...	*d*, (11,4), *b*, <u>(9,6)</u>, *a*	*d*, <u>(11,4)</u>, *b*	*d*, (11,4), *b*, <u>(9,6)</u>, *a*, (10,5), *c*	---

Therefore, the strengths of the strongest paths are:

	$P_D[*,a]$	$P_D[*,b]$	$P_D[*,c]$	$P_D[*,d]$
$P_D[a,*]$	---	(10,5)	(10,5)	(9,6)
$P_D[b,*]$	(9,6)	---	(9,6)	(9,6)
$P_D[c,*]$	(9,6)	(11,4)	---	(9,6)
$P_D[d,*]$	(9,6)	(11,4)	(9,6)	---

We get $\mathcal{O} = \{ab, ac, cb, db\}$ and $\mathcal{S} = \{a, d\}$.

Suppose, the strongest paths are calculated with the Floyd algorithm, as defined in section 2.3. Then the following table documents the 24 steps of the Floyd algorithm.

We start with

- $P_D[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(4,11)	(9,6)	(10,5)	b	a	$P_D[b,c]$ is updated from (4,11) to (9,6); $pred[b,c]$ is updated from b to a
2	a	b	d	(4,11)	(9,6)	(7,8)	b	a	$P_D[b,d]$ is updated from (4,11) to (7,8); $pred[b,d]$ is updated from b to a
3	a	c	b	(11,4)	(5,10)	(6,9)	c	a	
4	a	c	d	(9,6)	(5,10)	(7,8)	c	a	
5	a	d	b	(11,4)	(8,7)	(6,9)	d	a	
6	a	d	c	(6,9)	(8,7)	(10,5)	d	a	$P_D[d,c]$ is updated from (6,9) to (8,7); $pred[d,c]$ is updated from d to a
7	b	a	c	(10,5)	(6,9)	(9,6)	a	a	
8	b	a	d	(7,8)	(6,9)	(7,8)	a	a	
9	b	c	a	(5,10)	(11,4)	(9,6)	c	b	$P_D[c,a]$ is updated from (5,10) to (9,6); $pred[c,a]$ is updated from c to b
10	b	c	d	(9,6)	(11,4)	(7,8)	c	a	
11	b	d	a	(8,7)	(11,4)	(9,6)	d	b	$P_D[d,a]$ is updated from (8,7) to (9,6); $pred[d,a]$ is updated from d to b
12	b	d	c	(8,7)	(11,4)	(9,6)	a	a	$P_D[d,c]$ is updated from (8,7) to (9,6)
13	c	a	b	(6,9)	(10,5)	(11,4)	a	c	$P_D[a,b]$ is updated from (6,9) to (10,5); $pred[a,b]$ is updated from a to c
14	c	a	d	(7,8)	(10,5)	(9,6)	a	c	$P_D[a,d]$ is updated from (7,8) to (9,6); $pred[a,d]$ is updated from a to c
15	c	b	a	(9,6)	(9,6)	(9,6)	b	b	
16	c	b	d	(7,8)	(9,6)	(9,6)	a	c	$P_D[b,d]$ is updated from (7,8) to (9,6); $pred[b,d]$ is updated from a to c
17	c	d	a	(9,6)	(9,6)	(9,6)	b	b	
18	c	d	b	(11,4)	(9,6)	(11,4)	d	c	
19	d	a	b	(10,5)	(9,6)	(11,4)	c	d	
20	d	a	c	(10,5)	(9,6)	(9,6)	a	a	
21	d	b	a	(9,6)	(9,6)	(9,6)	b	b	
22	d	b	c	(9,6)	(9,6)	(9,6)	a	a	
23	d	c	a	(9,6)	(9,6)	(9,6)	b	b	
24	d	c	b	(11,4)	(9,6)	(11,4)	c	d	

3.5. Example 5

The basic idea for the following example has been proposed by Cretney (1998).

3.5.1. Situation #1

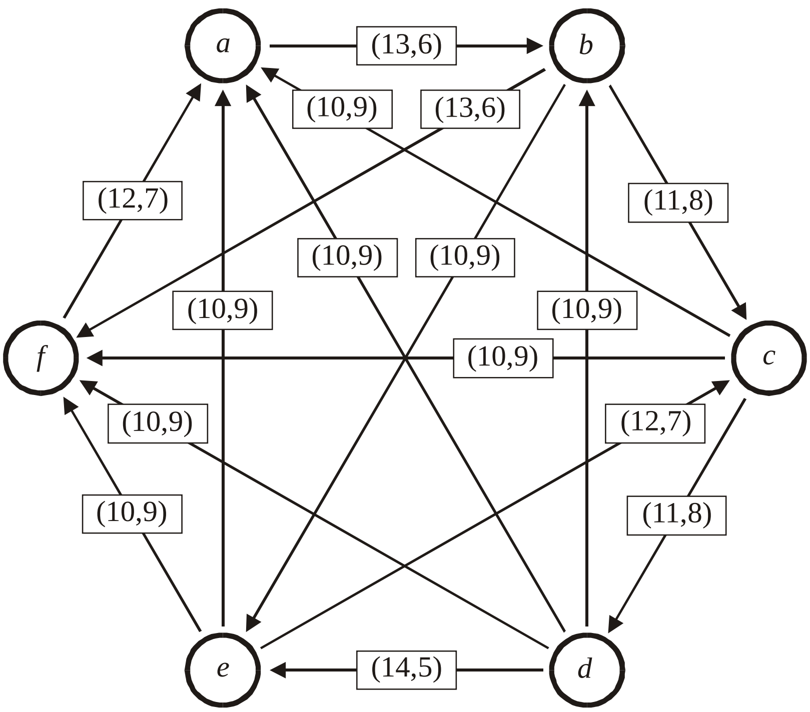
Example 5 (old):

3 voters	$a \succ_v d \succ_v e \succ_v b \succ_v c \succ_v f$
3 voters	$b \succ_v f \succ_v e \succ_v c \succ_v d \succ_v a$
4 voters	$c \succ_v a \succ_v b \succ_v f \succ_v d \succ_v e$
1 voter	$d \succ_v b \succ_v c \succ_v e \succ_v f \succ_v a$
4 voters	$d \succ_v e \succ_v f \succ_v a \succ_v b \succ_v c$
2 voters	$e \succ_v c \succ_v b \succ_v d \succ_v f \succ_v a$
2 voters	$f \succ_v a \succ_v c \succ_v d \succ_v b \succ_v e$

The pairwise matrix N^{old} looks as follows:

	$N^{old}[*,a]$	$N^{old}[*,b]$	$N^{old}[*,c]$	$N^{old}[*,d]$	$N^{old}[*,e]$	$N^{old}[*,f]$
$N^{old}[a,*]$	---	13	9	9	9	7
$N^{old}[b,*]$	6	---	11	9	10	13
$N^{old}[c,*]$	10	8	---	11	7	10
$N^{old}[d,*]$	10	10	8	---	14	10
$N^{old}[e,*]$	10	9	12	5	---	10
$N^{old}[f,*]$	12	6	9	9	9	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*	... to *e*	... to *f*
from *a* ...	---	*a*, (13,6), *b*	*a*, (13,6), *b*, (11,8), *c*	*a*, (13,6), *b*, (11,8), *c*, (11,8), *d*	*a*, (13,6), *b*, (11,8), *c*, (11,8), *d*, (14,5), *e*	*a*, (13,6), *b*, (13,6), *f*
from *b* ...	*b*, (13,6), *f*, (12,7), *a*	---	*b*, (11,8), *c*	*b*, (11,8), *c*, (11,8), *d*	*b*, (11,8), *c*, (11,8), *d*, (14,5), *e*	*b*, (13,6), *f*
from *c* ...	*c*, (10,9), *a*	*c*, (10,9), *a*, (13,6), *b*	---	*c*, (11,8), *d*	*c*, (11,8), *d*, (14,5), *e*	*c*, (10,9), *f*
from *d* ...	*d*, (10,9), *a*	*d*, (10,9), *b*	*d*, (14,5), *e*, (12,7), *c*	---	*d*, (14,5), *e*	*d*, (10,9), *f*
from *e* ...	*e*, (10,9), *a*	*e*, (10,9), *a*, (13,6), *b*	*e*, (12,7), *c*	*e*, (12,7), *c*, (11,8), *d*	---	*e*, (10,9), *f*
from *f* ...	*f*, (12,7), *a*	*f*, (12,7), *a*, (13,6), *b*	*f*, (12,7), *a*, (13,6), *b*, (11,8), *c*	*f*, (12,7), *a*, (13,6), *b*, (11,8), *c*, (11,8), *d*	*f*, (12,7), *a*, (13,6), *b*, (11,8), *c*, (11,8), *d*, (14,5), *e*	---

We get $O^{\text{old}} = \{ab, ac, ad, ae, af, bc, bd, be, bf, dc, de, ec, fc, fd, fe\}$ and $\mathcal{S}^{\text{old}} = \{a\}$.

Suppose, the strongest paths are calculated with the Floyd algorithm, as defined in section 2.3. Then the following table documents the 120 steps of the Floyd algorithm.

We start with

- $P_D[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(11,8)	(6,13)	(9,10)	b	a	
2	a	b	d	(9,10)	(6,13)	(9,10)	b	a	
3	a	b	e	(10,9)	(6,13)	(9,10)	b	a	
4	a	b	f	(13,6)	(6,13)	(7,12)	b	a	
5	a	c	b	(8,11)	(10,9)	(13,6)	c	a	$P_D[c,b]$ is updated from (8,11) to (10,9); $pred[c,b]$ is updated from c to a
6	a	c	d	(11,8)	(10,9)	(9,10)	c	a	
7	a	c	e	(7,12)	(10,9)	(9,10)	c	a	$P_D[c,e]$ is updated from (7,12) to (9,10); $pred[c,e]$ is updated from c to a
8	a	c	f	(10,9)	(10,9)	(7,12)	c	a	
9	a	d	b	(10,9)	(10,9)	(13,6)	d	a	
10	a	d	c	(8,11)	(10,9)	(9,10)	d	a	$P_D[d,c]$ is updated from (8,11) to (9,10); $pred[d,c]$ is updated from d to a
11	a	d	e	(14,5)	(10,9)	(9,10)	d	a	
12	a	d	f	(10,9)	(10,9)	(7,12)	d	a	
13	a	e	b	(9,10)	(10,9)	(13,6)	e	a	$P_D[e,b]$ is updated from (9,10) to (10,9); $pred[e,b]$ is updated from e to a
14	a	e	c	(12,7)	(10,9)	(9,10)	e	a	
15	a	e	d	(5,14)	(10,9)	(9,10)	e	a	$P_D[e,d]$ is updated from (5,14) to (9,10); $pred[e,d]$ is updated from e to a
16	a	e	f	(10,9)	(10,9)	(7,12)	e	a	
17	a	f	b	(6,13)	(12,7)	(13,6)	f	a	$P_D[f,b]$ is updated from (6,13) to (12,7); $pred[f,b]$ is updated from f to a
18	a	f	c	(9,10)	(12,7)	(9,10)	f	a	
19	a	f	d	(9,10)	(12,7)	(9,10)	f	a	
20	a	f	e	(9,10)	(12,7)	(9,10)	f	a	
21	b	a	c	(9,10)	(13,6)	(11,8)	a	b	$P_D[a,c]$ is updated from (9,10) to (11,8); $pred[a,c]$ is updated from a to b
22	b	a	d	(9,10)	(13,6)	(9,10)	a	b	
23	b	a	e	(9,10)	(13,6)	(10,9)	a	b	$P_D[a,e]$ is updated from (9,10) to (10,9); $pred[a,e]$ is updated from a to b
24	b	a	f	(7,12)	(13,6)	(13,6)	a	b	$P_D[a,f]$ is updated from (7,12) to (13,6); $pred[a,f]$ is updated from a to b
25	b	c	a	(10,9)	(10,9)	(6,13)	c	b	
26	b	c	d	(11,8)	(10,9)	(9,10)	c	b	
27	b	c	e	(9,10)	(10,9)	(10,9)	a	b	$P_D[c,e]$ is updated from (9,10) to (10,9); $pred[c,e]$ is updated from a to b
28	b	c	f	(10,9)	(10,9)	(13,6)	c	b	
29	b	d	a	(10,9)	(10,9)	(6,13)	d	b	
30	b	d	c	(9,10)	(10,9)	(11,8)	a	b	$P_D[d,c]$ is updated from (9,10) to (10,9); $pred[d,c]$ is updated from a to b

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
31	b	d	e	(14,5)	(10,9)	(10,9)	d	b	
32	b	d	f	(10,9)	(10,9)	(13,6)	d	b	
33	b	e	a	(10,9)	(10,9)	(6,13)	e	b	
34	b	e	c	(12,7)	(10,9)	(11,8)	e	b	
35	b	e	d	(9,10)	(10,9)	(9,10)	a	b	
36	b	e	f	(10,9)	(10,9)	(13,6)	e	b	
37	b	f	a	(12,7)	(12,7)	(6,13)	f	b	
38	b	f	c	(9,10)	(12,7)	(11,8)	f	b	$P_D[f,c]$ is updated from (9,10) to (11,8); $pred[f,c]$ is updated from f to b
39	b	f	d	(9,10)	(12,7)	(9,10)	f	b	
40	b	f	e	(9,10)	(12,7)	(10,9)	f	b	$P_D[f,e]$ is updated from (9,10) to (10,9); $pred[f,e]$ is updated from f to b
41	c	a	b	(13,6)	(11,8)	(10,9)	a	a	
42	c	a	d	(9,10)	(11,8)	(11,8)	a	c	$P_D[a,d]$ is updated from (9,10) to (11,8); $pred[a,d]$ is updated from a to c
43	c	a	e	(10,9)	(11,8)	(10,9)	b	b	
44	c	a	f	(13,6)	(11,8)	(10,9)	b	c	
45	c	b	a	(6,13)	(11,8)	(10,9)	b	c	$P_D[b,a]$ is updated from (6,13) to (10,9); $pred[b,a]$ is updated from b to c
46	c	b	d	(9,10)	(11,8)	(11,8)	b	c	$P_D[b,d]$ is updated from (9,10) to (11,8); $pred[b,d]$ is updated from b to c
47	c	b	e	(10,9)	(11,8)	(10,9)	b	b	
48	c	b	f	(13,6)	(11,8)	(10,9)	b	c	
49	c	d	a	(10,9)	(10,9)	(10,9)	d	c	
50	c	d	b	(10,9)	(10,9)	(10,9)	d	a	
51	c	d	e	(14,5)	(10,9)	(10,9)	d	b	
52	c	d	f	(10,9)	(10,9)	(10,9)	d	c	
53	c	e	a	(10,9)	(12,7)	(10,9)	e	c	
54	c	e	b	(10,9)	(12,7)	(10,9)	a	a	
55	c	e	d	(9,10)	(12,7)	(11,8)	a	c	$P_D[e,d]$ is updated from (9,10) to (11,8); $pred[e,d]$ is updated from a to c
56	c	e	f	(10,9)	(12,7)	(10,9)	e	c	
57	c	f	a	(12,7)	(11,8)	(10,9)	f	c	
58	c	f	b	(12,7)	(11,8)	(10,9)	a	a	
59	c	f	d	(9,10)	(11,8)	(11,8)	f	c	$P_D[f,d]$ is updated from (9,10) to (11,8); $pred[f,d]$ is updated from f to c
60	c	f	e	(10,9)	(11,8)	(10,9)	b	b	

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
61	d	a	b	(13,6)	(11,8)	(10,9)	a	d	
62	d	a	c	(11,8)	(11,8)	(10,9)	b	b	
63	d	a	e	(10,9)	(11,8)	(14,5)	b	d	$P_D[a,e]$ is updated from (10,9) to (11,8); $pred[a,e]$ is updated from b to d
64	d	a	f	(13,6)	(11,8)	(10,9)	b	d	
65	d	b	a	(10,9)	(11,8)	(10,9)	c	d	
66	d	b	c	(11,8)	(11,8)	(10,9)	b	b	
67	d	b	e	(10,9)	(11,8)	(14,5)	b	d	$P_D[b,e]$ is updated from (10,9) to (11,8); $pred[b,e]$ is updated from b to d
68	d	b	f	(13,6)	(11,8)	(10,9)	b	d	
69	d	c	a	(10,9)	(11,8)	(10,9)	c	d	
70	d	c	b	(10,9)	(11,8)	(10,9)	a	d	
71	d	c	e	(10,9)	(11,8)	(14,5)	b	d	$P_D[c,e]$ is updated from (10,9) to (11,8); $pred[c,e]$ is updated from b to d
72	d	c	f	(10,9)	(11,8)	(10,9)	c	d	
73	d	e	a	(10,9)	(11,8)	(10,9)	e	d	
74	d	e	b	(10,9)	(11,8)	(10,9)	a	d	
75	d	e	c	(12,7)	(11,8)	(10,9)	e	b	
76	d	e	f	(10,9)	(11,8)	(10,9)	e	d	
77	d	f	a	(12,7)	(11,8)	(10,9)	f	d	
78	d	f	b	(12,7)	(11,8)	(10,9)	a	d	
79	d	f	c	(11,8)	(11,8)	(10,9)	b	b	
80	d	f	e	(10,9)	(11,8)	(14,5)	b	d	$P_D[f,e]$ is updated from (10,9) to (11,8); $pred[f,e]$ is updated from b to d
81	e	a	b	(13,6)	(11,8)	(10,9)	a	a	
82	e	a	c	(11,8)	(11,8)	(12,7)	b	e	
83	e	a	d	(11,8)	(11,8)	(11,8)	c	c	
84	e	a	f	(13,6)	(11,8)	(10,9)	b	e	
85	e	b	a	(10,9)	(11,8)	(10,9)	c	e	
86	e	b	c	(11,8)	(11,8)	(12,7)	b	e	
87	e	b	d	(11,8)	(11,8)	(11,8)	c	c	
88	e	b	f	(13,6)	(11,8)	(10,9)	b	e	
89	e	c	a	(10,9)	(11,8)	(10,9)	c	e	
90	e	c	b	(10,9)	(11,8)	(10,9)	a	a	

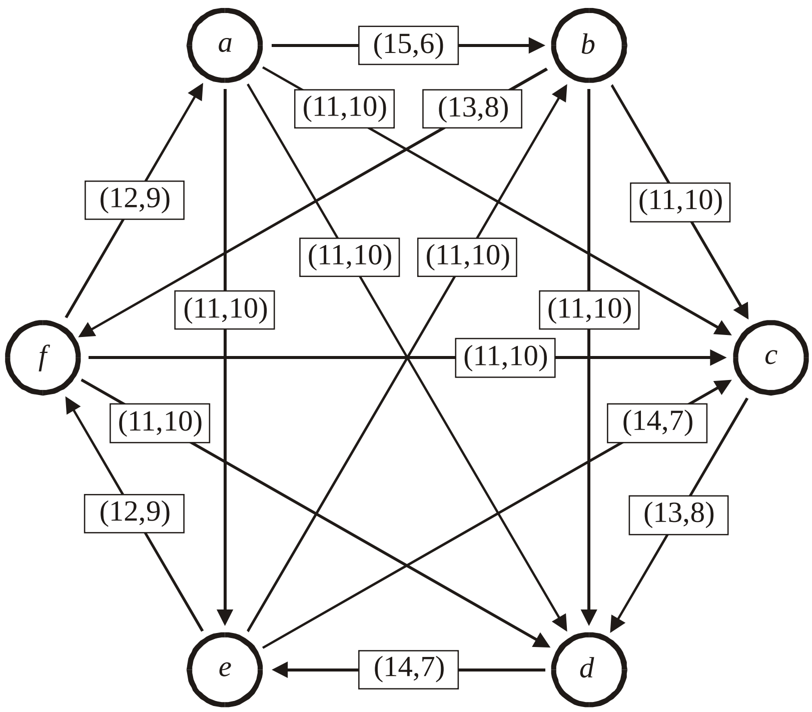
	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
91	e	c	d	(11,8)	(11,8)	(11,8)	c	c	
92	e	c	f	(10,9)	(11,8)	(10,9)	c	e	
93	e	d	a	(10,9)	(14,5)	(10,9)	d	e	
94	e	d	b	(10,9)	(14,5)	(10,9)	d	a	
95	e	d	c	(10,9)	(14,5)	(12,7)	b	e	$P_D[d,c]$ is updated from (10,9) to (12,7); $pred[d,c]$ is updated from b to e
96	e	d	f	(10,9)	(14,5)	(10,9)	d	e	
97	e	f	a	(12,7)	(11,8)	(10,9)	f	e	
98	e	f	b	(12,7)	(11,8)	(10,9)	a	a	
99	e	f	c	(11,8)	(11,8)	(12,7)	b	e	
100	e	f	d	(11,8)	(11,8)	(11,8)	c	c	
101	f	a	b	(13,6)	(13,6)	(12,7)	a	a	
102	f	a	c	(11,8)	(13,6)	(11,8)	b	b	
103	f	a	d	(11,8)	(13,6)	(11,8)	c	c	
104	f	a	e	(11,8)	(13,6)	(11,8)	d	d	
105	f	b	a	(10,9)	(13,6)	(12,7)	c	f	$P_D[b,a]$ is updated from (10,9) to (12,7); $pred[b,a]$ is updated from c to f
106	f	b	c	(11,8)	(13,6)	(11,8)	b	b	
107	f	b	d	(11,8)	(13,6)	(11,8)	c	c	
108	f	b	e	(11,8)	(13,6)	(11,8)	d	d	
109	f	c	a	(10,9)	(10,9)	(12,7)	c	f	
110	f	c	b	(10,9)	(10,9)	(12,7)	a	a	
111	f	c	d	(11,8)	(10,9)	(11,8)	c	c	
112	f	c	e	(11,8)	(10,9)	(11,8)	d	d	
113	f	d	a	(10,9)	(10,9)	(12,7)	d	f	
114	f	d	b	(10,9)	(10,9)	(12,7)	d	a	
115	f	d	c	(12,7)	(10,9)	(11,8)	e	b	
116	f	d	e	(14,5)	(10,9)	(11,8)	d	d	
117	f	e	a	(10,9)	(10,9)	(12,7)	e	f	
118	f	e	b	(10,9)	(10,9)	(12,7)	a	a	
119	f	e	c	(12,7)	(10,9)	(11,8)	e	b	
120	f	e	d	(11,8)	(10,9)	(11,8)	c	c	

3.5.2. Situation #2

When 2 $a \succ_v e \succ_v f \succ_v c \succ_v b \succ_v d$ ballots are added, then the pairwise matrix N^{new} looks as follows:

	$N^{new}[*,a]$	$N^{new}[*,b]$	$N^{new}[*,c]$	$N^{new}[*,d]$	$N^{new}[*,e]$	$N^{new}[*,f]$
$N^{new}[a,*]$	---	15	11	11	11	9
$N^{new}[b,*]$	6	---	11	11	10	13
$N^{new}[c,*]$	10	10	---	13	7	10
$N^{new}[d,*]$	10	10	8	---	14	10
$N^{new}[e,*]$	10	11	14	7	---	12
$N^{new}[f,*]$	12	8	11	11	9	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*	... to *e*	... to *f*
from *a* ...	---	*a*, (15,6), *b*	*a*, (11,10), *c*	*a*, (11,10), *d*	*a*, (11,10), *e*	*a*, (15,6), *b*, (13,8), *f*
from *b* ...	*b*, (13,8), *f*, (12,9), *a*	---	*b*, (11,10), *c*	*b*, (11,10), *d*	*b*, (11,10), *d*, (14,7), *e*	*b*, (13,8), *f*
from *c* ...	*c*, (13,8), *d*, (14,7), *e*, (12,9), *f*, (12,9), *a*	*c*, (13,8), *d*, (14,7), *e*, (12,9), *f*, (12,9), *a*, (15,6), *b*	---	*c*, (13,8), *d*	*c*, (13,8), *d*, (14,7), *e*	*c*, (13,8), *d*, (14,7), *e*, (12,9), *f*
from *d* ...	*d*, (14,7), *e*, (12,9), *f*, (12,9), *a*	*d*, (14,7), *e*, (12,9), *f*, (12,9), *a*, (15,6), *b*	*d*, (14,7), *e*, (14,7), *c*	---	*d*, (14,7), *e*	*d*, (14,7), *e*, (12,9), *f*
from *e* ...	*e*, (12,9), *f*, (12,9), *a*	*e*, (12,9), *f*, (12,9), *a*, (15,6), *b*	*e*, (14,7), *c*	*e*, (14,7), *c*, (13,8), *d*	---	*e*, (12,9), *f*
from *f* ...	*f*, (12,9), *a*	*f*, (12,9), *a*, (15,6), *b*	*f*, (11,10), *c*	*f*, (11,10), *d*	*f*, (12,9), *a*, (11,10), *e*	---

We get $\mathcal{O}^{new}$ = {*ab*, *af*, *bf*, *ca*, *cb*, *cf*, *da*, *db*, *dc*, *de*, *df*, *ea*, *eb*, *ec*, *ef*} and $\mathcal{S}^{new}$ = {*d*}.

Thus the 2 $a \succ_v e \succ_v f \succ_v c \succ_v b \succ_v d$ voters change the unique winner from alternative *a* to alternative *d*.

Suppose, the strongest paths are calculated with the Floyd algorithm, as defined in section 2.3. Then the following table documents the 120 steps of the Floyd algorithm.

We start with

- $P_D[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(11,10)	(6,15)	(11,10)	b	a	
2	a	b	d	(11,10)	(6,15)	(11,10)	b	a	
3	a	b	e	(10,11)	(6,15)	(11,10)	b	a	
4	a	b	f	(13,8)	(6,15)	(9,12)	b	a	
5	a	c	b	(10,11)	(10,11)	(15,6)	c	a	
6	a	c	d	(13,8)	(10,11)	(11,10)	c	a	
7	a	c	e	(7,14)	(10,11)	(11,10)	c	a	$P_D[c,e]$ is updated from (7,14) to (10,11); $pred[c,e]$ is updated from c to a
8	a	c	f	(10,11)	(10,11)	(9,12)	c	a	
9	a	d	b	(10,11)	(10,11)	(15,6)	d	a	
10	a	d	c	(8,13)	(10,11)	(11,10)	d	a	$P_D[d,c]$ is updated from (8,13) to (10,11); $pred[d,c]$ is updated from d to a
11	a	d	e	(14,7)	(10,11)	(11,10)	d	a	
12	a	d	f	(10,11)	(10,11)	(9,12)	d	a	
13	a	e	b	(11,10)	(10,11)	(15,6)	e	a	
14	a	e	c	(14,7)	(10,11)	(11,10)	e	a	
15	a	e	d	(7,14)	(10,11)	(11,10)	e	a	$P_D[e,d]$ is updated from (7,14) to (10,11); $pred[e,d]$ is updated from e to a
16	a	e	f	(12,9)	(10,11)	(9,12)	e	a	
17	a	f	b	(8,13)	(12,9)	(15,6)	f	a	$P_D[f,b]$ is updated from (8,13) to (12,9); $pred[f,b]$ is updated from f to a
18	a	f	c	(11,10)	(12,9)	(11,10)	f	a	
19	a	f	d	(11,10)	(12,9)	(11,10)	f	a	
20	a	f	e	(9,12)	(12,9)	(11,10)	f	a	$P_D[f,e]$ is updated from (9,12) to (11,10); $pred[f,e]$ is updated from f to a
21	b	a	c	(11,10)	(15,6)	(11,10)	a	b	
22	b	a	d	(11,10)	(15,6)	(11,10)	a	b	
23	b	a	e	(11,10)	(15,6)	(10,11)	a	b	
24	b	a	f	(9,12)	(15,6)	(13,8)	a	b	$P_D[a,f]$ is updated from (9,12) to (13,8); $pred[a,f]$ is updated from a to b
25	b	c	a	(10,11)	(10,11)	(6,15)	c	b	
26	b	c	d	(13,8)	(10,11)	(11,10)	c	b	
27	b	c	e	(10,11)	(10,11)	(10,11)	a	b	
28	b	c	f	(10,11)	(10,11)	(13,8)	c	b	
29	b	d	a	(10,11)	(10,11)	(6,15)	d	b	
30	b	d	c	(10,11)	(10,11)	(11,10)	a	b	

	i	*j*	*k*	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	*pred*[*j*,*k*]	*pred*[*i*,*k*]	result
31	*b*	*d*	*e*	(14,7)	(10,11)	(10,11)	*d*	*b*	
32	*b*	*d*	*f*	(10,11)	(10,11)	(13,8)	*d*	*b*	
33	*b*	*e*	*a*	(10,11)	(11,10)	(6,15)	*e*	*b*	
34	*b*	*e*	*c*	(14,7)	(11,10)	(11,10)	*e*	*b*	
35	*b*	*e*	*d*	(10,11)	(11,10)	(11,10)	*a*	*b*	$P_D[e,d]$ is updated from (10,11) to (11,10); *pred*[*e*,*d*] is updated from *a* to *b*
36	*b*	*e*	*f*	(12,9)	(11,10)	(13,8)	*e*	*b*	
37	*b*	*f*	*a*	(12,9)	(12,9)	(6,15)	*f*	*b*	
38	*b*	*f*	*c*	(11,10)	(12,9)	(11,10)	*f*	*b*	
39	*b*	*f*	*d*	(11,10)	(12,9)	(11,10)	*f*	*b*	
40	*b*	*f*	*e*	(11,10)	(12,9)	(10,11)	*a*	*b*	
41	*c*	*a*	*b*	(15,6)	(11,10)	(10,11)	*a*	*c*	
42	*c*	*a*	*d*	(11,10)	(11,10)	(13,8)	*a*	*c*	
43	*c*	*a*	*e*	(11,10)	(11,10)	(10,11)	*a*	*a*	
44	*c*	*a*	*f*	(13,8)	(11,10)	(10,11)	*b*	*c*	
45	*c*	*b*	*a*	(6,15)	(11,10)	(10,11)	*b*	*c*	$P_D[b,a]$ is updated from (6,15) to (10,11); *pred*[*b*,*a*] is updated from *b* to *c*
46	*c*	*b*	*d*	(11,10)	(11,10)	(13,8)	*b*	*c*	
47	*c*	*b*	*e*	(10,11)	(11,10)	(10,11)	*b*	*a*	
48	*c*	*b*	*f*	(13,8)	(11,10)	(10,11)	*b*	*c*	
49	*c*	*d*	*a*	(10,11)	(10,11)	(10,11)	*d*	*c*	
50	*c*	*d*	*b*	(10,11)	(10,11)	(10,11)	*d*	*c*	
51	*c*	*d*	*e*	(14,7)	(10,11)	(10,11)	*d*	*a*	
52	*c*	*d*	*f*	(10,11)	(10,11)	(10,11)	*d*	*c*	
53	*c*	*e*	*a*	(10,11)	(14,7)	(10,11)	*e*	*c*	
54	*c*	*e*	*b*	(11,10)	(14,7)	(10,11)	*e*	*c*	
55	*c*	*e*	*d*	(11,10)	(14,7)	(13,8)	*b*	*c*	$P_D[e,d]$ is updated from (11,10) to (13,8); *pred*[*e*,*d*] is updated from *b* to *c*
56	*c*	*e*	*f*	(12,9)	(14,7)	(10,11)	*e*	*c*	
57	*c*	*f*	*a*	(12,9)	(11,10)	(10,11)	*f*	*c*	
58	*c*	*f*	*b*	(12,9)	(11,10)	(10,11)	*a*	*c*	
59	*c*	*f*	*d*	(11,10)	(11,10)	(13,8)	*f*	*c*	
60	*c*	*f*	*e*	(11,10)	(11,10)	(10,11)	*a*	*a*	

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
61	d	a	b	(15,6)	(11,10)	(10,11)	a	d	
62	d	a	c	(11,10)	(11,10)	(10,11)	a	a	
63	d	a	e	(11,10)	(11,10)	(14,7)	a	d	
64	d	a	f	(13,8)	(11,10)	(10,11)	b	d	
65	d	b	a	(10,11)	(11,10)	(10,11)	c	d	
66	d	b	c	(11,10)	(11,10)	(10,11)	b	a	
67	d	b	e	(10,11)	(11,10)	(14,7)	b	d	$P_D[b,e]$ is updated from (10,11) to (11,10); $pred[b,e]$ is updated from b to d
68	d	b	f	(13,8)	(11,10)	(10,11)	b	d	
69	d	c	a	(10,11)	(13,8)	(10,11)	c	d	
70	d	c	b	(10,11)	(13,8)	(10,11)	c	d	
71	d	c	e	(10,11)	(13,8)	(14,7)	a	d	$P_D[c,e]$ is updated from (10,11) to (13,8); $pred[c,e]$ is updated from a to d
72	d	c	f	(10,11)	(13,8)	(10,11)	c	d	
73	d	e	a	(10,11)	(13,8)	(10,11)	e	d	
74	d	e	b	(11,10)	(13,8)	(10,11)	e	d	
75	d	e	c	(14,7)	(13,8)	(10,11)	e	a	
76	d	e	f	(12,9)	(13,8)	(10,11)	e	d	
77	d	f	a	(12,9)	(11,10)	(10,11)	f	d	
78	d	f	b	(12,9)	(11,10)	(10,11)	a	d	
79	d	f	c	(11,10)	(11,10)	(10,11)	f	a	
80	d	f	e	(11,10)	(11,10)	(14,7)	a	d	
81	e	a	b	(15,6)	(11,10)	(11,10)	a	e	
82	e	a	c	(11,10)	(11,10)	(14,7)	a	e	
83	e	a	d	(11,10)	(11,10)	(13,8)	a	c	
84	e	a	f	(13,8)	(11,10)	(12,9)	b	e	
85	e	b	a	(10,11)	(11,10)	(10,11)	c	e	
86	e	b	c	(11,10)	(11,10)	(14,7)	b	e	
87	e	b	d	(11,10)	(11,10)	(13,8)	b	c	
88	e	b	f	(13,8)	(11,10)	(12,9)	b	e	
89	e	c	a	(10,11)	(13,8)	(10,11)	c	e	
90	e	c	b	(10,11)	(13,8)	(11,10)	c	e	$P_D[c,b]$ is updated from (10,11) to (11,10); $pred[c,b]$ is updated from c to e

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
91	e	c	d	(13,8)	(13,8)	(13,8)	c	c	
92	e	c	f	(10,11)	(13,8)	(12,9)	c	e	$P_D[c,f]$ is updated from (10,11) to (12,9); $pred[c,f]$ is updated from c to e
93	e	d	a	(10,11)	(14,7)	(10,11)	d	e	
94	e	d	b	(10,11)	(14,7)	(11,10)	d	e	$P_D[d,b]$ is updated from (10,11) to (11,10); $pred[d,b]$ is updated from d to e
95	e	d	c	(10,11)	(14,7)	(14,7)	a	e	$P_D[d,c]$ is updated from (10,11) to (14,7); $pred[d,c]$ is updated from a to e
96	e	d	f	(10,11)	(14,7)	(12,9)	d	e	$P_D[d,f]$ is updated from (10,11) to (12,9); $pred[d,f]$ is updated from d to e
97	e	f	a	(12,9)	(11,10)	(10,11)	f	e	
98	e	f	b	(12,9)	(11,10)	(11,10)	a	e	
99	e	f	c	(11,10)	(11,10)	(14,7)	f	e	
100	e	f	d	(11,10)	(11,10)	(13,8)	f	c	
101	f	a	b	(15,6)	(13,8)	(12,9)	a	a	
102	f	a	c	(11,10)	(13,8)	(11,10)	a	f	
103	f	a	d	(11,10)	(13,8)	(11,10)	a	f	
104	f	a	e	(11,10)	(13,8)	(11,10)	a	a	
105	f	b	a	(10,11)	(13,8)	(12,9)	c	f	$P_D[b,a]$ is updated from (10,11) to (12,9); $pred[b,a]$ is updated from c to f
106	f	b	c	(11,10)	(13,8)	(11,10)	b	f	
107	f	b	d	(11,10)	(13,8)	(11,10)	b	f	
108	f	b	e	(11,10)	(13,8)	(11,10)	d	a	
109	f	c	a	(10,11)	(12,9)	(12,9)	c	f	$P_D[c,a]$ is updated from (10,11) to (12,9); $pred[c,a]$ is updated from c to f
110	f	c	b	(11,10)	(12,9)	(12,9)	e	a	$P_D[c,b]$ is updated from (11,10) to (12,9); $pred[c,b]$ is updated from e to a
111	f	c	d	(13,8)	(12,9)	(11,10)	c	f	
112	f	c	e	(13,8)	(12,9)	(11,10)	d	a	
113	f	d	a	(10,11)	(12,9)	(12,9)	d	f	$P_D[d,a]$ is updated from (10,11) to (12,9); $pred[d,a]$ is updated from d to f
114	f	d	b	(11,10)	(12,9)	(12,9)	e	a	$P_D[d,b]$ is updated from (11,10) to (12,9); $pred[d,b]$ is updated from e to a
115	f	d	c	(14,7)	(12,9)	(11,10)	e	f	
116	f	d	e	(14,7)	(12,9)	(11,10)	d	a	
117	f	e	a	(10,11)	(12,9)	(12,9)	e	f	$P_D[e,a]$ is updated from (10,11) to (12,9); $pred[e,a]$ is updated from e to f
118	f	e	b	(11,10)	(12,9)	(12,9)	e	a	$P_D[e,b]$ is updated from (11,10) to (12,9); $pred[e,b]$ is updated from e to a
119	f	e	c	(14,7)	(12,9)	(11,10)	e	f	
120	f	e	d	(13,8)	(12,9)	(11,10)	c	f	

3.6. Example 6

Suppose an alternative e is added with $N[d,e] > 0$ and $N[e,d] = 0$ for at least one already running alternative d. Then *independence from Pareto-dominated alternatives* (IPDA) says that we must get:

(3.6.1) $\forall\, x,y \in A \setminus \{e\}: xy \in \mathcal{O}^{\text{old}} \Leftrightarrow xy \in \mathcal{O}^{\text{new}}$.

(3.6.2) $\forall\, x \in A \setminus \{e\}: x \in \mathcal{S}^{\text{old}} \Leftrightarrow x \in \mathcal{S}^{\text{new}}$.

The following example demonstrates that the Schulze method, as defined in section 2.2, does not satisfy IPDA. This example has been proposed by Eppley (2003).

3.6.1. Situation #1

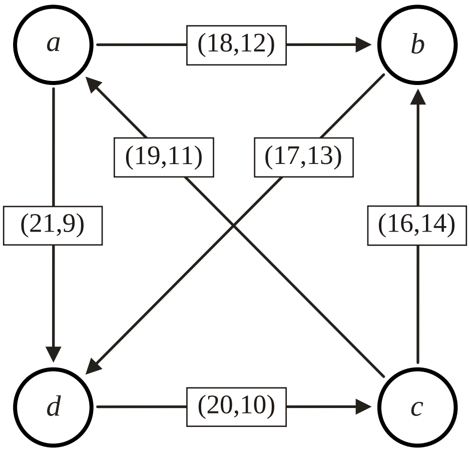
Example 6 (old):

3 voters	$a \succ_v b \succ_v d \succ_v c$
5 voters	$a \succ_v d \succ_v b \succ_v c$
1 voter	$a \succ_v d \succ_v c \succ_v b$
2 voters	$b \succ_v a \succ_v d \succ_v c$
2 voters	$b \succ_v d \succ_v c \succ_v a$
4 voters	$c \succ_v a \succ_v b \succ_v d$
6 voters	$c \succ_v b \succ_v a \succ_v d$
2 voters	$d \succ_v b \succ_v c \succ_v a$
5 voters	$d \succ_v c \succ_v a \succ_v b$

The pairwise matrix N^{old} looks as follows:

	$N^{\text{old}}[*,a]$	$N^{\text{old}}[*,b]$	$N^{\text{old}}[*,c]$	$N^{\text{old}}[*,d]$
$N^{\text{old}}[a,*]$	---	18	11	21
$N^{\text{old}}[b,*]$	12	---	14	17
$N^{\text{old}}[c,*]$	19	16	---	10
$N^{\text{old}}[d,*]$	9	13	20	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*
from *a* ...	---	*a*, <u>(18,12)</u>, *b*	*a*, (21,9), *d*, <u>(20,10)</u>, *c*	*a*, <u>(21,9)</u>, *d*
from *b* ...	*b*, <u>(17,13)</u>, *d*, (20,10), *c*, (19,11), *a*	---	*b*, <u>(17,13)</u>, *d*, (20,10), *c*	*b*, <u>(17,13)</u>, *d*
from *c* ...	*c*, <u>(19,11)</u>, *a*	*c*, (19,11), *a*, <u>(18,12)</u>, *b*	---	*c*, <u>(19,11)</u>, *a*, (21,9), *d*
from *d* ...	*d*, (20,10), *c*, <u>(19,11)</u>, *a*	*d*, (20,10), *c*, (19,11), *a*, <u>(18,12)</u>, *b*	*d*, <u>(20,10)</u>, *c*	---

We get $\mathcal{O}^{\text{old}} = \{ab, ac, ad, cb, db, dc\}$ and $\mathcal{S}^{\text{old}} = \{a\}$.

Suppose, the strongest paths are calculated with the Floyd algorithm, as defined in section 2.3. Then the following table documents the 24 steps of the Floyd algorithm.

We start with

- $P_D[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	*j*	*k*	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	*pred*[*j*,*k*]	*pred*[*i*,*k*]	result
1	*a*	*b*	*c*	(14,16)	(12,18)	(11,19)	*b*	*a*	
2	*a*	*b*	*d*	(17,13)	(12,18)	(21,9)	*b*	*a*	
3	*a*	*c*	*b*	(16,14)	(19,11)	(18,12)	*c*	*a*	$P_D[c,b]$ is updated from (16,14) to (18,12); *pred*[*c*,*b*] is updated from *c* to *a*
4	*a*	*c*	*d*	(10,20)	(19,11)	(21,9)	*c*	*a*	$P_D[c,d]$ is updated from (10,20) to (19,11); *pred*[*c*,*d*] is updated from *c* to *a*
5	*a*	*d*	*b*	(13,17)	(9,21)	(18,12)	*d*	*a*	
6	*a*	*d*	*c*	(20,10)	(9,21)	(11,19)	*d*	*a*	
7	*b*	*a*	*c*	(11,19)	(18,12)	(14,16)	*a*	*b*	$P_D[a,c]$ is updated from (11,19) to (14,16); *pred*[*a*,*c*] is updated from *a* to *b*
8	*b*	*a*	*d*	(21,9)	(18,12)	(17,13)	*a*	*b*	
9	*b*	*c*	*a*	(19,11)	(18,12)	(12,18)	*c*	*b*	
10	*b*	*c*	*d*	(19,11)	(18,12)	(17,13)	*a*	*b*	
11	*b*	*d*	*a*	(9,21)	(13,17)	(12,18)	*d*	*b*	$P_D[d,a]$ is updated from (9,21) to (12,18); *pred*[*d*,*a*] is updated from *d* to *b*
12	*b*	*d*	*c*	(20,10)	(13,17)	(14,16)	*d*	*b*	
13	*c*	*a*	*b*	(18,12)	(14,16)	(18,12)	*a*	*a*	
14	*c*	*a*	*d*	(21,9)	(14,16)	(19,11)	*a*	*a*	
15	*c*	*b*	*a*	(12,18)	(14,16)	(19,11)	*b*	*c*	$P_D[b,a]$ is updated from (12,18) to (14,16); *pred*[*b*,*a*] is updated from *b* to *c*
16	*c*	*b*	*d*	(17,13)	(14,16)	(19,11)	*b*	*a*	
17	*c*	*d*	*a*	(12,18)	(20,10)	(19,11)	*b*	*c*	$P_D[d,a]$ is updated from (12,18) to (19,11); *pred*[*d*,*a*] is updated from *b* to *c*
18	*c*	*d*	*b*	(13,17)	(20,10)	(18,12)	*d*	*a*	$P_D[d,b]$ is updated from (13,17) to (18,12); *pred*[*d*,*b*] is updated from *d* to *a*
19	*d*	*a*	*b*	(18,12)	(21,9)	(18,12)	*a*	*a*	
20	*d*	*a*	*c*	(14,16)	(21,9)	(20,10)	*b*	*d*	$P_D[a,c]$ is updated from (14,16) to (20,10); *pred*[*a*,*c*] is updated from *b* to *d*
21	*d*	*b*	*a*	(14,16)	(17,13)	(19,11)	*c*	*c*	$P_D[b,a]$ is updated from (14,16) to (17,13)
22	*d*	*b*	*c*	(14,16)	(17,13)	(20,10)	*b*	*d*	$P_D[b,c]$ is updated from (14,16) to (17,13); *pred*[*b*,*c*] is updated from *b* to *d*
23	*d*	*c*	*a*	(19,11)	(19,11)	(19,11)	*c*	*c*	
24	*d*	*c*	*b*	(18,12)	(19,11)	(18,12)	*a*	*a*	

3.6.2. Situation #2

Suppose alternative *e* is added as follows:

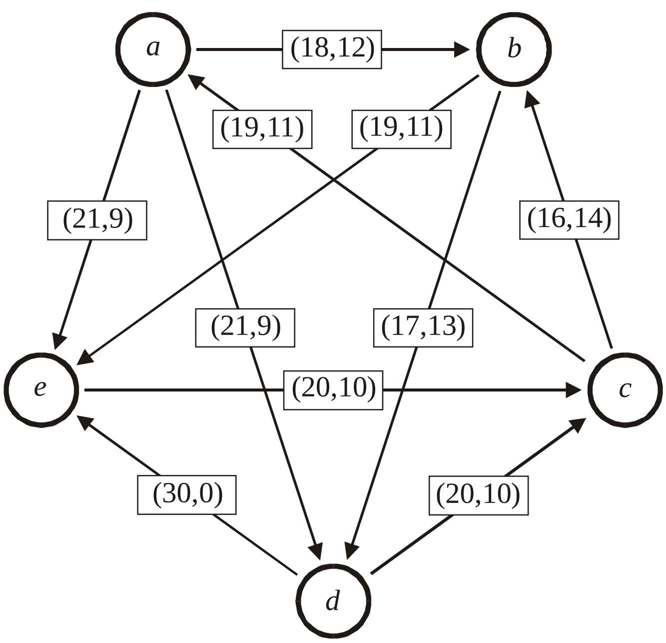
Example 6 (new):

3 voters	$a \succ_v b \succ_v d \succ_v e \succ_v c$
5 voters	$a \succ_v d \succ_v e \succ_v b \succ_v c$
1 voter	$a \succ_v d \succ_v e \succ_v c \succ_v b$
2 voters	$b \succ_v a \succ_v d \succ_v e \succ_v c$
2 voters	$b \succ_v d \succ_v e \succ_v c \succ_v a$
4 voters	$c \succ_v a \succ_v b \succ_v d \succ_v e$
6 voters	$c \succ_v b \succ_v a \succ_v d \succ_v e$
2 voters	$d \succ_v b \succ_v e \succ_v c \succ_v a$
5 voters	$d \succ_v e \succ_v c \succ_v a \succ_v b$

The pairwise matrix N^{new} looks as follows:

	$N^{new}[*,a]$	$N^{new}[*,b]$	$N^{new}[*,c]$	$N^{new}[*,d]$	$N^{new}[*,e]$
$N^{new}[a,*]$	---	18	11	21	21
$N^{new}[b,*]$	12	---	14	17	19
$N^{new}[c,*]$	19	16	---	10	10
$N^{new}[d,*]$	9	13	20	---	30
$N^{new}[e,*]$	9	11	20	0	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*	... to *e*
from *a* ...	---	*a*, <u>(18,12)</u>, *b*	*a*, (21,9), *d*, <u>(20,10)</u>, *c*	*a*, <u>(21,9)</u>, *d*	*a*, <u>(21,9)</u>, *e*
from *b* ...	*b*, <u>(19,11)</u>, *e*, (20,10), *c*, <u>(19,11)</u>, *a*	---	*b*, <u>(19,11)</u>, *e*, (20,10), *c*	*b*, <u>(19,11)</u>, *e*, (20,10), *c*, <u>(19,11)</u>, *a*, (21,9), *d*	*b*, <u>(19,11)</u>, *e*
from *c* ...	*c*, <u>(19,11)</u>, *a*	*c*, (19,11), *a*, <u>(18,12)</u>, *b*	---	*c*, <u>(19,11)</u>, *a*, (21,9), *d*	*c*, <u>(19,11)</u>, *a*, (21,9), *e*
from *d* ...	*d*, (20,10), *c*, <u>(19,11)</u>, *a*	*d*, (20,10), *c*, (19,11), *a*, <u>(18,12)</u>, *b*	*d*, <u>(20,10)</u>, *c*	---	*d*, <u>(30,0)</u>, *e*
from *e* ...	*e*, (20,10), *c*, <u>(19,11)</u>, *a*	*e*, (20,10), *c*, (19,11), *a*, <u>(18,12)</u>, *b*	*e*, <u>(20,10)</u>, *c*	*e*, (20,10), *c*, <u>(19,11)</u>, *a*, (21,9), *d*	---

We get $\mathcal{O}^{new}$ = {*ac*, *ad*, *ae*, *ba*, *bc*, *bd*, *be*, *dc*, *de*, *ec*} and $\mathcal{S}^{new}$ = {*b*}.

Suppose, the strongest paths are calculated with the Floyd algorithm, as defined in section 2.3. Then the following table documents the 60 steps of the Floyd algorithm.

We start with

- $P_D[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(14,16)	(12,18)	(11,19)	b	a	
2	a	b	d	(17,13)	(12,18)	(21,9)	b	a	
3	a	b	e	(19,11)	(12,18)	(21,9)	b	a	
4	a	c	b	(16,14)	(19,11)	(18,12)	c	a	$P_D[c,b]$ is updated from (16,14) to (18,12); $pred[c,b]$ is updated from c to a
5	a	c	d	(10,20)	(19,11)	(21,9)	c	a	$P_D[c,d]$ is updated from (10,20) to (19,11); $pred[c,d]$ is updated from c to a
6	a	c	e	(10,20)	(19,11)	(21,9)	c	a	$P_D[c,e]$ is updated from (10,20) to (19,11); $pred[c,e]$ is updated from c to a
7	a	d	b	(13,17)	(9,21)	(18,12)	d	a	
8	a	d	c	(20,10)	(9,21)	(11,19)	d	a	
9	a	d	e	(30,0)	(9,21)	(21,9)	d	a	
10	a	e	b	(11,19)	(9,21)	(18,12)	e	a	
11	a	e	c	(20,10)	(9,21)	(11,19)	e	a	
12	a	e	d	(0,30)	(9,21)	(21,9)	e	a	$P_D[e,d]$ is updated from (0,30) to (9,21); $pred[e,d]$ is updated from e to a
13	b	a	c	(11,19)	(18,12)	(14,16)	a	b	$P_D[a,c]$ is updated from (11,19) to (14,16); $pred[a,c]$ is updated from a to b
14	b	a	d	(21,9)	(18,12)	(17,13)	a	b	
15	b	a	e	(21,9)	(18,12)	(19,11)	a	b	
16	b	c	a	(19,11)	(18,12)	(12,18)	c	b	
17	b	c	d	(19,11)	(18,12)	(17,13)	a	b	
18	b	c	e	(19,11)	(18,12)	(19,11)	a	b	
19	b	d	a	(9,21)	(13,17)	(12,18)	d	b	$P_D[d,a]$ is updated from (9,21) to (12,18); $pred[d,a]$ is updated from d to b
20	b	d	c	(20,10)	(13,17)	(14,16)	d	b	
21	b	d	e	(30,0)	(13,17)	(19,11)	d	b	
22	b	e	a	(9,21)	(11,19)	(12,18)	e	b	$P_D[e,a]$ is updated from (9,21) to (11,19); $pred[e,a]$ is updated from e to b
23	b	e	c	(20,10)	(11,19)	(14,16)	e	b	
24	b	e	d	(9,21)	(11,19)	(17,13)	a	b	$P_D[e,d]$ is updated from (9,21) to (11,19); $pred[e,d]$ is updated from a to b
25	c	a	b	(18,12)	(14,16)	(18,12)	a	a	
26	c	a	d	(21,9)	(14,16)	(19,11)	a	a	
27	c	a	e	(21,9)	(14,16)	(19,11)	a	a	
28	c	b	a	(12,18)	(14,16)	(19,11)	b	c	$P_D[b,a]$ is updated from (12,18) to (14,16); $pred[b,a]$ is updated from b to c
29	c	b	d	(17,13)	(14,16)	(19,11)	b	a	
30	c	b	e	(19,11)	(14,16)	(19,11)	b	a	

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
31	c	d	a	(12,18)	(20,10)	(19,11)	b	c	$P_D[d,a]$ is updated from (12,18) to (19,11); $pred[d,a]$ is updated from b to c
32	c	d	b	(13,17)	(20,10)	(18,12)	d	a	$P_D[d,b]$ is updated from (13,17) to (18,12); $pred[d,b]$ is updated from d to a
33	c	d	e	(30,0)	(20,10)	(19,11)	d	a	
34	c	e	a	(11,19)	(20,10)	(19,11)	b	c	$P_D[e,a]$ is updated from (11,19) to (19,11); $pred[e,a]$ is updated from b to c
35	c	e	b	(11,19)	(20,10)	(18,12)	e	a	$P_D[e,b]$ is updated from (11,19) to (18,12); $pred[e,b]$ is updated from e to a
36	c	e	d	(11,19)	(20,10)	(19,11)	b	a	$P_D[e,d]$ is updated from (11,19) to (19,11); $pred[e,d]$ is updated from b to a
37	d	a	b	(18,12)	(21,9)	(18,12)	a	a	
38	d	a	c	(14,16)	(21,9)	(20,10)	b	d	$P_D[a,c]$ is updated from (14,16) to (20,10); $pred[a,c]$ is updated from b to d
39	d	a	e	(21,9)	(21,9)	(30,0)	a	d	
40	d	b	a	(14,16)	(17,13)	(19,11)	c	c	$P_D[b,a]$ is updated from (14,16) to (17,13)
41	d	b	c	(14,16)	(17,13)	(20,10)	b	d	$P_D[b,c]$ is updated from (14,16) to (17,13); $pred[b,c]$ is updated from b to d
42	d	b	e	(19,11)	(17,13)	(30,0)	b	d	
43	d	c	a	(19,11)	(19,11)	(19,11)	c	c	
44	d	c	b	(18,12)	(19,11)	(18,12)	a	a	
45	d	c	e	(19,11)	(19,11)	(30,0)	a	d	
46	d	e	a	(19,11)	(19,11)	(19,11)	c	c	
47	d	e	b	(18,12)	(19,11)	(18,12)	a	a	
48	d	e	c	(20,10)	(19,11)	(20,10)	e	d	
49	e	a	b	(18,12)	(21,9)	(18,12)	a	a	
50	e	a	c	(20,10)	(21,9)	(20,10)	d	e	
51	e	a	d	(21,9)	(21,9)	(19,11)	a	a	
52	e	b	a	(17,13)	(19,11)	(19,11)	c	c	$P_D[b,a]$ is updated from (17,13) to (19,11)
53	e	b	c	(17,13)	(19,11)	(20,10)	d	e	$P_D[b,c]$ is updated from (17,13) to (19,11); $pred[b,c]$ is updated from d to e
54	e	b	d	(17,13)	(19,11)	(19,11)	b	a	$P_D[b,d]$ is updated from (17,13) to (19,11); $pred[b,d]$ is updated from b to a
55	e	c	a	(19,11)	(19,11)	(19,11)	c	c	
56	e	c	b	(18,12)	(19,11)	(18,12)	a	a	
57	e	c	d	(19,11)	(19,11)	(19,11)	a	a	
58	e	d	a	(19,11)	(30,0)	(19,11)	c	c	
59	e	d	b	(18,12)	(30,0)	(18,12)	a	a	
60	e	d	c	(20,10)	(30,0)	(20,10)	d	e	

3.7. Example 7

When each voter $v \in V$ casts a linear order $\succ_v$ on A, then all definitions for $\succ_D$, that satisfy presumption (2.1.1), are equivalent. However, when some voters cast non-linear orders, then there are many possible definitions for the strength of a link. The following example illustrates how the different definitions for the strength of a link can lead to different winners.

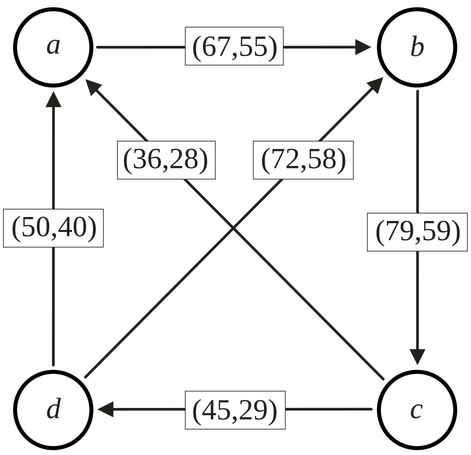
Example 7:

6	voters	$a \succ_v b \succ_v c \succ_v d$
8	voters	$a \approx_v b \succ_v c \approx_v d$
8	voters	$a \approx_v c \succ_v b \approx_v d$
18	voters	$a \approx_v c \succ_v d \succ_v b$
8	voters	$a \approx_v c \approx_v d \succ_v b$
40	voters	$b \succ_v a \approx_v c \approx_v d$
4	voters	$c \succ_v b \succ_v d \succ_v a$
9	voters	$c \succ_v d \succ_v a \succ_v b$
8	voters	$c \approx_v d \succ_v a \approx_v b$
14	voters	$d \succ_v a \succ_v b \succ_v c$
11	voters	$d \succ_v b \succ_v c \succ_v a$
4	voters	$d \succ_v c \succ_v a \succ_v b$

The pairwise matrix N looks as follows:

	$N[*,a]$	$N[*,b]$	$N[*,c]$	$N[*,d]$
$N[a,*]$	---	67	28	40
$N[b,*]$	55	---	79	58
$N[c,*]$	36	59	---	45
$N[d,*]$	50	72	29	---

The corresponding digraph looks as follows:



a) margin

We get: $(N[b,c],N[c,b]) \succ_{margin} (N[c,d],N[d,c]) \succ_{margin} (N[d,b],N[b,d]) \succ_{margin} (N[a,b],N[b,a]) \succ_{margin} (N[d,a],N[a,d]) \succ_{margin} (N[c,a],N[a,c])$.

The pairwise victories are:

bc with a margin of $N[b,c] – N[c,b] = 20$
cd with a margin of $N[c,d] – N[d,c] = 16$
db with a margin of $N[d,b] – N[b,d] = 14$
ab with a margin of $N[a,b] – N[b,a] = 12$
da with a margin of $N[d,a] – N[a,d] = 10$
ca with a margin of $N[c,a] – N[a,c] = 8$

The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*
from *a* ...	---	*a*, $\underline{(67,55)}$, *b*	*a*, $\underline{(67,55)}$, *b*, (79,59), *c*	*a*, $\underline{(67,55)}$, *b*, (79,59), *c*, (45,29), *d*
from *b* ...	*b*, (79,59), *c*, (45,29), *d*, $\underline{(50,40)}$, *a*	---	*b*, $\underline{(79,59)}$, *c*	*b*, (79,59), *c*, $\underline{(45,29)}$, *d*
from *c* ...	*c*, (45,29), *d*, $\underline{(50,40)}$, *a*	*c*, (45,29), *d*, $\underline{(72,58)}$, *b*	---	*c*, $\underline{(45,29)}$, *d*
from *d* ...	*d*, $\underline{(50,40)}$, *a*	*d*, $\underline{(72,58)}$, *b*	*d*, $\underline{(72,58)}$, *b*, (79,59), *c*	---

We get $\mathcal{O}_{margin} = \{ab, ac, ad, bc, bd, cd\}$ and $\mathcal{S}_{margin} = \{a\}$.

Suppose, the strongest paths are calculated with the Floyd algorithm, as defined in section 2.3. Then the following table documents the 24 steps of the Floyd algorithm.

We start with

- $P_{margin}[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_{margin}[j,k]$	$P_{margin}[j,i]$	$P_{margin}[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(79,59)	(55,67)	(28,36)	b	a	
2	a	b	d	(58,72)	(55,67)	(40,50)	b	a	$P_{margin}[b,d]$ is updated from (58,72) to (55,67); $pred[b,d]$ is updated from b to a
3	a	c	b	(59,79)	(36,28)	(67,55)	c	a	$P_{margin}[c,b]$ is updated from (59,79) to (36,28); $pred[c,b]$ is updated from c to a
4	a	c	d	(45,29)	(36,28)	(40,50)	c	a	
5	a	d	b	(72,58)	(50,40)	(67,55)	d	a	
6	a	d	c	(29,45)	(50,40)	(28,36)	d	a	$P_{margin}[d,c]$ is updated from (29,45) to (28,36); $pred[d,c]$ is updated from d to a
7	b	a	c	(28,36)	(67,55)	(79,59)	a	b	$P_{margin}[a,c]$ is updated from (28,36) to (67,55); $pred[a,c]$ is updated from a to b
8	b	a	d	(40,50)	(67,55)	(55,67)	a	a	
9	b	c	a	(36,28)	(36,28)	(55,67)	c	b	
10	b	c	d	(45,29)	(36,28)	(55,67)	c	a	
11	b	d	a	(50,40)	(72,58)	(55,67)	d	b	
12	b	d	c	(28,36)	(72,58)	(79,59)	a	b	$P_{margin}[d,c]$ is updated from (28,36) to (72,58); $pred[d,c]$ is updated from a to b
13	c	a	b	(67,55)	(67,55)	(36,28)	a	a	
14	c	a	d	(40,50)	(67,55)	(45,29)	a	c	$P_{margin}[a,d]$ is updated from (40,50) to (67,55); $pred[a,d]$ is updated from a to c
15	c	b	a	(55,67)	(79,59)	(36,28)	b	c	$P_{margin}[b,a]$ is updated from (55,67) to (36,28); $pred[b,a]$ is updated from b to c
16	c	b	d	(55,67)	(79,59)	(45,29)	a	c	$P_{margin}[b,d]$ is updated from (55,67) to (45,29); $pred[b,d]$ is updated from a to c
17	c	d	a	(50,40)	(72,58)	(36,28)	d	c	
18	c	d	b	(72,58)	(72,58)	(36,28)	d	a	
19	d	a	b	(67,55)	(67,55)	(72,58)	a	d	
20	d	a	c	(67,55)	(67,55)	(72,58)	b	b	
21	d	b	a	(36,28)	(45,29)	(50,40)	c	d	$P_{margin}[b,a]$ is updated from (36,28) to (50,40); $pred[b,a]$ is updated from c to d
22	d	b	c	(79,59)	(45,29)	(72,58)	b	b	
23	d	c	a	(36,28)	(45,29)	(50,40)	c	d	$P_{margin}[c,a]$ is updated from (36,28) to (50,40); $pred[c,a]$ is updated from c to d
24	d	c	b	(36,28)	(45,29)	(72,58)	a	d	$P_{margin}[c,b]$ is updated from (36,28) to (72,58); $pred[c,b]$ is updated from a to d

b) ratio

We get: $(N[c,d],N[d,c]) \succ_{ratio} (N[b,c],N[c,b]) \succ_{ratio} (N[c,a],N[a,c]) \succ_{ratio} (N[d,a],N[a,d]) \succ_{ratio} (N[d,b],N[b,d]) \succ_{ratio} (N[a,b],N[b,a])$.

The pairwise victories are:

cd with a ratio of $N[c,d] / N[d,c] = 1.552$
bc with a ratio of $N[b,c] / N[c,b] = 1.339$
ca with a ratio of $N[c,a] / N[a,c] = 1.286$
da with a ratio of $N[d,a] / N[a,d] = 1.250$
db with a ratio of $N[d,b] / N[b,d] = 1.241$
ab with a ratio of $N[a,b] / N[b,a] = 1.218$

The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*
from *a* ...	---	*a*, <u>(67,55)</u>, *b*	*a*, <u>(67,55)</u>, *b*, (79,59), *c*	*a*, <u>(67,55)</u>, *b*, (79,59), *c*, (45,29), *d*
from *b* ...	*b*, (79,59), *c*, <u>(36,28)</u>, *a*	---	*b*, <u>(79,59)</u>, *c*	*b*, <u>(79,59)</u>, *c*, (45,29), *d*
from *c* ...	*c*, <u>(36,28)</u>, *a*	*c*, (45,29), *d*, <u>(72,58)</u>, *b*	---	*c*, <u>(45,29)</u>, *d*
from *d* ...	*d*, <u>(50,40)</u>, *a*	*d*, <u>(72,58)</u>, *b*	*d*, <u>(72,58)</u>, *b*, (79,59), *c*	---

We get $\mathcal{O}_{ratio} = \{ba, bc, bd, ca, cd, da\}$ and $\mathcal{S}_{ratio} = \{b\}$.

Suppose, the strongest paths are calculated with the Floyd algorithm, as defined in section 2.3. Then the following table documents the 24 steps of the Floyd algorithm.

We start with

- $P_{ratio}[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_{ratio}[j,k]$	$P_{ratio}[j,i]$	$P_{ratio}[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(79,59)	(55,67)	(28,36)	b	a	
2	a	b	d	(58,72)	(55,67)	(40,50)	b	a	
3	a	c	b	(59,79)	(36,28)	(67,55)	c	a	$P_{ratio}[c,b]$ is updated from (59,79) to (67,55); $pred[c,b]$ is updated from c to a
4	a	c	d	(45,29)	(36,28)	(40,50)	c	a	
5	a	d	b	(72,58)	(50,40)	(67,55)	d	a	
6	a	d	c	(29,45)	(50,40)	(28,36)	d	a	$P_{ratio}[d,c]$ is updated from (29,45) to (28,36); $pred[d,c]$ is updated from d to a
7	b	a	c	(28,36)	(67,55)	(79,59)	a	b	$P_{ratio}[a,c]$ is updated from (28,36) to (67,55); $pred[a,c]$ is updated from a to b
8	b	a	d	(40,50)	(67,55)	(58,72)	a	b	$P_{ratio}[a,d]$ is updated from (40,50) to (58,72); $pred[a,d]$ is updated from a to b
9	b	c	a	(36,28)	(67,55)	(55,67)	c	b	
10	b	c	d	(45,29)	(67,55)	(58,72)	c	b	
11	b	d	a	(50,40)	(72,58)	(55,67)	d	b	
12	b	d	c	(28,36)	(72,58)	(79,59)	a	b	$P_{ratio}[d,c]$ is updated from (28,36) to (72,58); $pred[d,c]$ is updated from a to b
13	c	a	b	(67,55)	(67,55)	(67,55)	a	a	
14	c	a	d	(58,72)	(67,55)	(45,29)	b	c	$P_{ratio}[a,d]$ is updated from (58,72) to (67,55); $pred[a,d]$ is updated from b to c
15	c	b	a	(55,67)	(79,59)	(36,28)	b	c	$P_{ratio}[b,a]$ is updated from (55,67) to (36,28); $pred[b,a]$ is updated from b to c
16	c	b	d	(58,72)	(79,59)	(45,29)	b	c	$P_{ratio}[b,d]$ is updated from (58,72) to (79,59); $pred[b,d]$ is updated from b to c
17	c	d	a	(50,40)	(72,58)	(36,28)	d	c	
18	c	d	b	(72,58)	(72,58)	(67,55)	d	a	
19	d	a	b	(67,55)	(67,55)	(72,58)	a	d	
20	d	a	c	(67,55)	(67,55)	(72,58)	b	b	
21	d	b	a	(36,28)	(79,59)	(50,40)	c	d	
22	d	b	c	(79,59)	(79,59)	(72,58)	b	b	
23	d	c	a	(36,28)	(45,29)	(50,40)	c	d	
24	d	c	b	(67,55)	(45,29)	(72,58)	a	d	$P_{ratio}[c,b]$ is updated from (67,55) to (72,58); $pred[c,b]$ is updated from a to d

c) winning votes

We get: $(N[b,c],N[c,b]) \succ_{win} (N[d,b],N[b,d]) \succ_{win} (N[a,b],N[b,a]) \succ_{win} (N[d,a],N[a,d]) \succ_{win} (N[c,d],N[d,c]) \succ_{win} (N[c,a],N[a,c])$.

The pairwise victories are:

bc with a support of $N[b,c] = 79$
db with a support of $N[d,b] = 72$
ab with a support of $N[a,b] = 67$
da with a support of $N[d,a] = 50$
cd with a support of $N[c,d] = 45$
ca with a support of $N[c,a] = 36$

The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*
from *a* ...	---	*a*, <u>(67,55)</u>, *b*	*a*, <u>(67,55)</u>, *b*, (79,59), *c*	*a*, (67,55), *b*, (79,59), *c*, <u>(45,29)</u>, *d*
from *b* ...	*b*, (79,59), *c*, <u>(45,29)</u>, *d*, (50,40), *a*	---	*b*, <u>(79,59)</u>, *c*	*b*, (79,59), *c*, <u>(45,29)</u>, *d*
from *c* ...	*c*, <u>(45,29)</u>, *d*, (50,40), *a*	*c*, <u>(45,29)</u>, *d*, (72,58), *b*	---	*c*, <u>(45,29)</u>, *d*
from *d* ...	*d*, <u>(50,40)</u>, *a*	*d*, <u>(72,58)</u>, *b*	*d*, <u>(72,58)</u>, *b*, (79,59), *c*	---

We get $\mathcal{O}_{win} = \{ab, ac, bc, da, db, dc\}$ and $\mathcal{S}_{win} = \{d\}$.

Suppose, the strongest paths are calculated with the Floyd algorithm, as defined in section 2.3. Then the following table documents the 24 steps of the Floyd algorithm.

We start with

- $P_{win}[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	*j*	*k*	$P_{win}[j,k]$	$P_{win}[j,i]$	$P_{win}[i,k]$	*pred*[*j*,*k*]	*pred*[*i*,*k*]	result
1	*a*	*b*	*c*	(79,59)	(55,67)	(28,36)	*b*	*a*	
2	*a*	*b*	*d*	(58,72)	(55,67)	(40,50)	*b*	*a*	$P_{win}[b,d]$ is updated from (58,72) to (55,67); *pred*[*b*,*d*] is updated from *b* to *a*
3	*a*	*c*	*b*	(59,79)	(36,28)	(67,55)	*c*	*a*	$P_{win}[c,b]$ is updated from (59,79) to (36,28); *pred*[*c*,*b*] is updated from *c* to *a*
4	*a*	*c*	*d*	(45,29)	(36,28)	(40,50)	*c*	*a*	
5	*a*	*d*	*b*	(72,58)	(50,40)	(67,55)	*d*	*a*	
6	*a*	*d*	*c*	(29,45)	(50,40)	(28,36)	*d*	*a*	$P_{win}[d,c]$ is updated from (29,45) to (28,36); *pred*[*d*,*c*] is updated from *d* to *a*
7	*b*	*a*	*c*	(28,36)	(67,55)	(79,59)	*a*	*b*	$P_{win}[a,c]$ is updated from (28,36) to (67,55); *pred*[*a*,*c*] is updated from *a* to *b*
8	*b*	*a*	*d*	(40,50)	(67,55)	(55,67)	*a*	*a*	
9	*b*	*c*	*a*	(36,28)	(36,28)	(55,67)	*c*	*b*	
10	*b*	*c*	*d*	(45,29)	(36,28)	(55,67)	*c*	*a*	
11	*b*	*d*	*a*	(50,40)	(72,58)	(55,67)	*d*	*b*	
12	*b*	*d*	*c*	(28,36)	(72,58)	(79,59)	*a*	*b*	$P_{win}[d,c]$ is updated from (28,36) to (72,58); *pred*[*d*,*c*] is updated from *a* to *b*
13	*c*	*a*	*b*	(67,55)	(67,55)	(36,28)	*a*	*a*	
14	*c*	*a*	*d*	(40,50)	(67,55)	(45,29)	*a*	*c*	$P_{win}[a,d]$ is updated from (40,50) to (45,29); *pred*[*a*,*d*] is updated from *a* to *c*
15	*c*	*b*	*a*	(55,67)	(79,59)	(36,28)	*b*	*c*	$P_{win}[b,a]$ is updated from (55,67) to (36,28); *pred*[*b*,*a*] is updated from *b* to *c*
16	*c*	*b*	*d*	(55,67)	(79,59)	(45,29)	*a*	*c*	$P_{win}[b,d]$ is updated from (55,67) to (45,29); *pred*[*b*,*d*] is updated from *a* to *c*
17	*c*	*d*	*a*	(50,40)	(72,58)	(36,28)	*d*	*c*	
18	*c*	*d*	*b*	(72,58)	(72,58)	(36,28)	*d*	*a*	
19	*d*	*a*	*b*	(67,55)	(45,29)	(72,58)	*a*	*d*	
20	*d*	*a*	*c*	(67,55)	(45,29)	(72,58)	*b*	*b*	
21	*d*	*b*	*a*	(36,28)	(45,29)	(50,40)	*c*	*d*	$P_{win}[b,a]$ is updated from (36,28) to (45,29); *pred*[*b*,*a*] is updated from *c* to *d*
22	*d*	*b*	*c*	(79,59)	(45,29)	(72,58)	*b*	*b*	
23	*d*	*c*	*a*	(36,28)	(45,29)	(50,40)	*c*	*d*	$P_{win}[c,a]$ is updated from (36,28) to (45,29); *pred*[*c*,*a*] is updated from *c* to *d*
24	*d*	*c*	*b*	(36,28)	(45,29)	(72,58)	*a*	*d*	$P_{win}[c,b]$ is updated from (36,28) to (45,29); *pred*[*c*,*b*] is updated from *a* to *d*

d) losing votes

We get: $(N[c,a],N[a,c]) \succ_{los} (N[c,d],N[d,c]) \succ_{los} (N[d,a],N[a,d]) \succ_{los} (N[a,b],N[b,a]) \succ_{los} (N[d,b],N[b,d]) \succ_{los} (N[b,c],N[c,b])$.

The pairwise victories are:

ca with an opposition of $N[a,c] = 28$
cd with an opposition of $N[d,c] = 29$
da with an opposition of $N[a,d] = 40$
ab with an opposition of $N[b,a] = 55$
db with an opposition of $N[b,d] = 58$
bc with an opposition of $N[c,b] = 59$

The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*
from *a* ...	---	*a*, <u>(67,55)</u>, *b*	*a*, (67,55), *b*, <u>(79,59)</u>, *c*	*a*, (67,55), *b*, <u>(79,59)</u>, *c*, (45,29), *d*
from *b* ...	*b*, <u>(79,59)</u>, *c*, (36,28), *a*	---	*b*, <u>(79,59)</u>, *c*	*b*, <u>(79,59)</u>, *c*, (45,29), *d*
from *c* ...	*c*, <u>(36,28)</u>, *a*	*c*, (36,28), *a*, <u>(67,55)</u>, *b*	---	*c*, <u>(45,29)</u>, *d*
from *d* ...	*d*, <u>(50,40)</u>, *a*	*d*, (50,40), *a* <u>(67,55)</u>, *b*	*d*, (50,40), *a* (67,55), *b*, <u>(79,59)</u>, *c*	---

We get $\mathcal{O}_{los} = \{ab, ca, cb, cd, da, db\}$ and $\mathcal{S}_{los} = \{c\}$.

Suppose, the strongest paths are calculated with the Floyd algorithm, as defined in section 2.3. Then the following table documents the 24 steps of the Floyd algorithm.

We start with

- $P_{los}[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_{los}[j,k]$	$P_{los}[j,i]$	$P_{los}[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(79,59)	(55,67)	(28,36)	b	a	
2	a	b	d	(58,72)	(55,67)	(40,50)	b	a	
3	a	c	b	(59,79)	(36,28)	(67,55)	c	a	$P_{los}[c,b]$ is updated from (59,79) to (67,55); $pred[c,b]$ is updated from c to a
4	a	c	d	(45,29)	(36,28)	(40,50)	c	a	
5	a	d	b	(72,58)	(50,40)	(67,55)	d	a	$P_{los}[d,b]$ is updated from (72,58) to (67,55); $pred[d,b]$ is updated from d to a
6	a	d	c	(29,45)	(50,40)	(28,36)	d	a	
7	b	a	c	(28,36)	(67,55)	(79,59)	a	b	$P_{los}[a,c]$ is updated from (28,36) to (79,59); $pred[a,c]$ is updated from a to b
8	b	a	d	(40,50)	(67,55)	(58,72)	a	b	$P_{los}[a,d]$ is updated from (40,50) to (58,72); $pred[a,d]$ is updated from a to b
9	b	c	a	(36,28)	(67,55)	(55,67)	c	b	
10	b	c	d	(45,29)	(67,55)	(58,72)	c	b	
11	b	d	a	(50,40)	(67,55)	(55,67)	d	b	
12	b	d	c	(29,45)	(67,55)	(79,59)	d	b	$P_{los}[d,c]$ is updated from (29,45) to (79,59); $pred[d,c]$ is updated from d to b
13	c	a	b	(67,55)	(79,59)	(67,55)	a	a	
14	c	a	d	(58,72)	(79,59)	(45,29)	b	c	$P_{los}[a,d]$ is updated from (58,72) to (79,59); $pred[a,d]$ is updated from b to c
15	c	b	a	(55,67)	(79,59)	(36,28)	b	c	$P_{los}[b,a]$ is updated from (55,67) to (79,59); $pred[b,a]$ is updated from b to c
16	c	b	d	(58,72)	(79,59)	(45,29)	b	c	$P_{los}[b,d]$ is updated from (58,72) to (79,59); $pred[b,d]$ is updated from b to c
17	c	d	a	(50,40)	(79,59)	(36,28)	d	c	
18	c	d	b	(67,55)	(79,59)	(67,55)	a	a	
19	d	a	b	(67,55)	(79,59)	(67,55)	a	a	
20	d	a	c	(79,59)	(79,59)	(79,59)	b	b	
21	d	b	a	(79,59)	(79,59)	(50,40)	c	d	
22	d	b	c	(79,59)	(79,59)	(79,59)	b	b	
23	d	c	a	(36,28)	(45,29)	(50,40)	c	d	
24	d	c	b	(67,55)	(45,29)	(67,55)	a	a	

3.8. Example 8

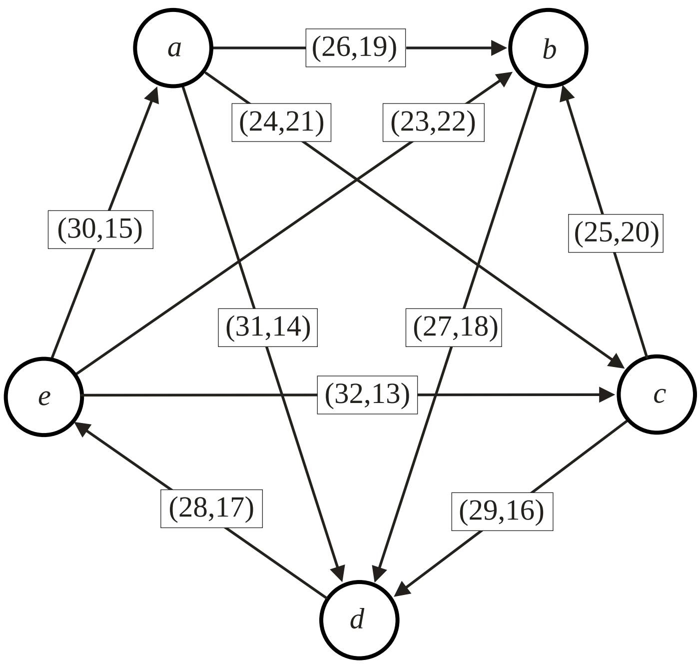
Example 8:

9	voters	$a \succ_v d \succ_v b \succ_v e \succ_v c$
6	voters	$b \succ_v c \succ_v a \succ_v d \succ_v e$
5	voters	$b \succ_v c \succ_v d \succ_v e \succ_v a$
2	voters	$c \succ_v d \succ_v b \succ_v e \succ_v a$
6	voters	$d \succ_v e \succ_v c \succ_v b \succ_v a$
14	voters	$e \succ_v a \succ_v c \succ_v b \succ_v d$
2	voters	$e \succ_v c \succ_v a \succ_v b \succ_v d$
1	voter	$e \succ_v d \succ_v a \succ_v c \succ_v b$

The pairwise matrix N looks as follows:

	$N[*,a]$	$N[*,b]$	$N[*,c]$	$N[*,d]$	$N[*,e]$
$N[a,*]$	---	26	24	31	15
$N[b,*]$	19	---	20	27	22
$N[c,*]$	21	25	---	29	13
$N[d,*]$	14	18	16	---	28
$N[e,*]$	30	23	32	17	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*	... to *e*
from *a* ...	---	*a*, (26,19), *b*	*a*, (31,14), *d*, (28,17), *e*, (32,13), *c*	*a*, (31,14), *d*	*a*, (31,14), *d*, (28,17), *e*
from *b* ...	*b*, (27,18), *d*, (28,17), *e*, (30,15), *a*	---	*b*, (27,18), *d*, (28,17), *e*, (32,13), *c*	*b*, (27,18), *d*	*b*, (27,18), *d*, (28,17), *e*
from *c* ...	*c*, (29,16), *d*, (28,17), *e*, (30,15), *a*	*c*, (29,16), *d*, (28,17), *e*, (30,15), *a*, (26,19), *b*	---	*c*, (29,16), *d*	*c*, (29,16), *d*, (28,17), *e*
from *d* ...	*d*, (28,17), *e*, (30,15), *a*	*d*, (28,17), *e*, (30,15), *a*, (26,19), *b*	*d*, (28,17), *e*, (32,13), *c*	---	*d*, (28,17), *e*
from *e* ...	*e*, (30,15), *a*	*e*, (30,15), *a*, (26,19), *b*	*e*, (32,13), *c*	*e*, (30,15), *a*, (31,14), *d*	---

We get $\mathcal{O}$ = {*ad*, *ba*, *bc*, *bd*, *be*, *cd*, *ea*, *ec*, *ed*} and $\mathcal{S}$ = {*b*}.

Suppose, the strongest paths are calculated with the Floyd algorithm, as defined in section 2.3. Then the following table documents the 60 steps of the Floyd algorithm.

We start with

- $P_D[i,j]$: = $(N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- *pred*$[i,j]$: = i for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(20,25)	(19,26)	(24,21)	b	a	
2	a	b	d	(27,18)	(19,26)	(31,14)	b	a	
3	a	b	e	(22,23)	(19,26)	(15,30)	b	a	
4	a	c	b	(25,20)	(21,24)	(26,19)	c	a	
5	a	c	d	(29,16)	(21,24)	(31,14)	c	a	
6	a	c	e	(13,32)	(21,24)	(15,30)	c	a	$P_D[c,e]$ is updated from (13,32) to (15,30); $pred[c,e]$ is updated from c to a
7	a	d	b	(18,27)	(14,31)	(26,19)	d	a	
8	a	d	c	(16,29)	(14,31)	(24,21)	d	a	
9	a	d	e	(28,17)	(14,31)	(15,30)	d	a	
10	a	e	b	(23,22)	(30,15)	(26,19)	e	a	$P_D[e,b]$ is updated from (23,22) to (26,19); $pred[e,b]$ is updated from e to a
11	a	e	c	(32,13)	(30,15)	(24,21)	e	a	
12	a	e	d	(17,28)	(30,15)	(31,14)	e	a	$P_D[e,d]$ is updated from (17,28) to (30,15); $pred[e,d]$ is updated from e to a
13	b	a	c	(24,21)	(26,19)	(20,25)	a	b	
14	b	a	d	(31,14)	(26,19)	(27,18)	a	b	
15	b	a	e	(15,30)	(26,19)	(22,23)	a	b	$P_D[a,e]$ is updated from (15,30) to (22,23); $pred[a,e]$ is updated from a to b
16	b	c	a	(21,24)	(25,20)	(19,26)	c	b	
17	b	c	d	(29,16)	(25,20)	(27,18)	c	b	
18	b	c	e	(15,30)	(25,20)	(22,23)	a	b	$P_D[c,e]$ is updated from (15,30) to (22,23); $pred[c,e]$ is updated from a to b
19	b	d	a	(14,31)	(18,27)	(19,26)	d	b	$P_D[d,a]$ is updated from (14,31) to (18,27); $pred[d,a]$ is updated from d to b
20	b	d	c	(16,29)	(18,27)	(20,25)	d	b	$P_D[d,c]$ is updated from (16,29) to (18,27); $pred[d,c]$ is updated from d to b
21	b	d	e	(28,17)	(18,27)	(22,23)	d	b	
22	b	e	a	(30,15)	(26,19)	(19,26)	e	b	
23	b	e	c	(32,13)	(26,19)	(20,25)	e	b	
24	b	e	d	(30,15)	(26,19)	(27,18)	a	b	
25	c	a	b	(26,19)	(24,21)	(25,20)	a	c	
26	c	a	d	(31,14)	(24,21)	(29,16)	a	c	
27	c	a	e	(22,23)	(24,21)	(22,23)	b	b	
28	c	b	a	(19,26)	(20,25)	(21,24)	b	c	$P_D[b,a]$ is updated from (19,26) to (20,25); $pred[b,a]$ is updated from b to c
29	c	b	d	(27,18)	(20,25)	(29,16)	b	c	
30	c	b	e	(22,23)	(20,25)	(22,23)	b	b	

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
31	c	d	a	(18,27)	(18,27)	(21,24)	b	c	
32	c	d	b	(18,27)	(18,27)	(25,20)	d	c	
33	c	d	e	(28,17)	(18,27)	(22,23)	d	b	
34	c	e	a	(30,15)	(32,13)	(21,24)	e	c	
35	c	e	b	(26,19)	(32,13)	(25,20)	a	c	
36	c	e	d	(30,15)	(32,13)	(29,16)	a	c	
37	d	a	b	(26,19)	(31,14)	(18,27)	a	d	
38	d	a	c	(24,21)	(31,14)	(18,27)	a	b	
39	d	a	e	(22,23)	(31,14)	(28,17)	b	d	$P_D[a,e]$ is updated from (22,23) to (28,17); $pred[a,e]$ is updated from b to d
40	d	b	a	(20,25)	(27,18)	(18,27)	c	b	
41	d	b	c	(20,25)	(27,18)	(18,27)	b	b	
42	d	b	e	(22,23)	(27,18)	(28,17)	b	d	$P_D[b,e]$ is updated from (22,23) to (27,18); $pred[b,e]$ is updated from b to d
43	d	c	a	(21,24)	(29,16)	(18,27)	c	b	
44	d	c	b	(25,20)	(29,16)	(18,27)	c	d	
45	d	c	e	(22,23)	(29,16)	(28,17)	b	d	$P_D[c,e]$ is updated from (22,23) to (28,17); $pred[c,e]$ is updated from b to d
46	d	e	a	(30,15)	(30,15)	(18,27)	e	b	
47	d	e	b	(26,19)	(30,15)	(18,27)	a	d	
48	d	e	c	(32,13)	(30,15)	(18,27)	e	b	
49	e	a	b	(26,19)	(28,17)	(26,19)	a	a	
50	e	a	c	(24,21)	(28,17)	(32,13)	a	e	$P_D[a,c]$ is updated from (24,21) to (28,17); $pred[a,c]$ is updated from a to e
51	e	a	d	(31,14)	(28,17)	(30,15)	a	a	
52	e	b	a	(20,25)	(27,18)	(30,15)	c	e	$P_D[b,a]$ is updated from (20,25) to (27,18); $pred[b,a]$ is updated from c to e
53	e	b	c	(20,25)	(27,18)	(32,13)	b	e	$P_D[b,c]$ is updated from (20,25) to (27,18); $pred[b,c]$ is updated from b to e
54	e	b	d	(27,18)	(27,18)	(30,15)	b	a	
55	e	c	a	(21,24)	(28,17)	(30,15)	c	e	$P_D[c,a]$ is updated from (21,24) to (28,17); $pred[c,a]$ is updated from c to e
56	e	c	b	(25,20)	(28,17)	(26,19)	c	a	$P_D[c,b]$ is updated from (25,20) to (26,19); $pred[c,b]$ is updated from c to a
57	e	c	d	(29,16)	(28,17)	(30,15)	c	a	
58	e	d	a	(18,27)	(28,17)	(30,15)	b	e	$P_D[d,a]$ is updated from (18,27) to (28,17); $pred[d,a]$ is updated from b to e
59	e	d	b	(18,27)	(28,17)	(26,19)	d	a	$P_D[d,b]$ is updated from (18,27) to (26,19); $pred[d,b]$ is updated from d to a
60	e	d	c	(18,27)	(28,17)	(32,13)	b	e	$P_D[d,c]$ is updated from (18,27) to (28,17); $pred[d,c]$ is updated from b to e

3.9. Example 9

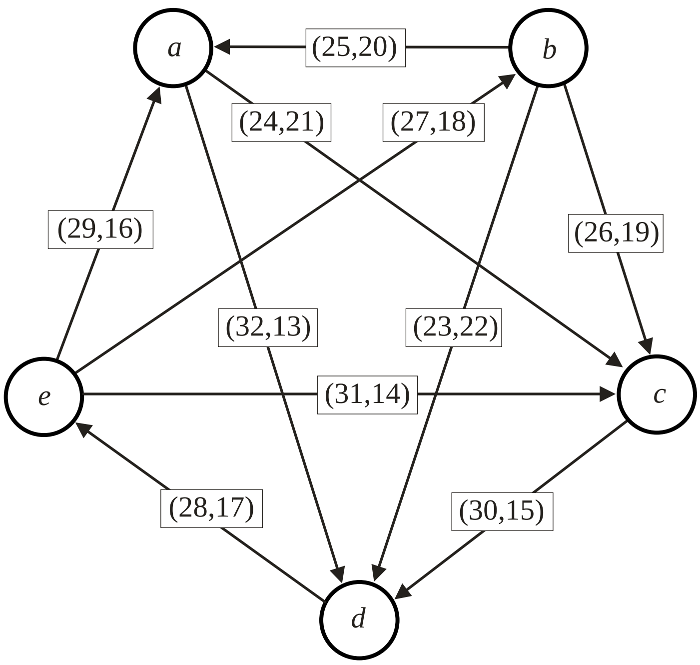
Example 9:

9	voters	$a \succ_v d \succ_v b \succ_v e \succ_v c$
1	voter	$b \succ_v a \succ_v c \succ_v e \succ_v d$
6	voters	$c \succ_v b \succ_v a \succ_v d \succ_v e$
2	voters	$c \succ_v d \succ_v b \succ_v e \succ_v a$
5	voters	$c \succ_v d \succ_v e \succ_v a \succ_v b$
6	voters	$d \succ_v e \succ_v c \succ_v a \succ_v b$
14	voters	$e \succ_v b \succ_v a \succ_v c \succ_v d$
2	voters	$e \succ_v b \succ_v c \succ_v a \succ_v d$

The pairwise matrix N looks as follows:

	$N[*,a]$	$N[*,b]$	$N[*,c]$	$N[*,d]$	$N[*,e]$
$N[a,*]$	---	20	24	32	16
$N[b,*]$	25	---	26	23	18
$N[c,*]$	21	19	---	30	14
$N[d,*]$	13	22	15	---	28
$N[e,*]$	29	27	31	17	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*	... to *e*
from *a* ...	---	*a*, (32,13), *d*, (28,17), *e*, (27,18), *b*	*a*, (32,13), *d*, (28,17), *e*, (31,14), *c*	*a*, (32,13), *d*	*a*, (32,13), *d*, (28,17), *e*
from *b* ...	*b*, (26,19), *c*, (30,15), *d*, (28,17), *e*, (29,16), *a*	---	*b*, (26,19), *c*	*b*, (26,19), *c*, (30,15), *d*	*b*, (26,19), *c*, (30,15), *d*, (28,17), *e*
from *c* ...	*c*, (30,15), *d*, (28,17), *e*, (29,16), *a*	*c*, (30,15), *d*, (28,17), *e*, (27,18), *b*	---	*c*, (30,15), *d*	*c*, (30,15), *d*, (28,17), *e*
from *d* ...	*d*, (28,17), *e*, (29,16), *a*	*d*, (28,17), *e*, (27,18), *b*	*d*, (28,17), *e*, (31,14), *c*	---	*d*, (28,17), *e*
from *e* ...	*e*, (29,16), *a*	*e*, (27,18), *b*	*e*, (31,14), *c*	*e*, (31,14), *c*, (30,15), *d*	---

We get $\mathcal{O}$ = {*ab*, *ad*, *cb*, *cd*, *db*, *ea*, *eb*, *ec*, *ed*} and $\mathcal{S}$ = {*e*}.

Suppose, the strongest paths are calculated with the Floyd algorithm, as defined in section 2.3. Then the following table documents the 60 steps of the Floyd algorithm.

We start with

- $P_D[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(26,19)	(25,20)	(24,21)	b	a	
2	a	b	d	(23,22)	(25,20)	(32,13)	b	a	$P_D[b,d]$ is updated from (23,22) to (25,20); $pred[b,d]$ is updated from b to a
3	a	b	e	(18,27)	(25,20)	(16,29)	b	a	
4	a	c	b	(19,26)	(21,24)	(20,25)	c	a	$P_D[c,b]$ is updated from (19,26) to (20,25); $pred[c,b]$ is updated from c to a
5	a	c	d	(30,15)	(21,24)	(32,13)	c	a	
6	a	c	e	(14,31)	(21,24)	(16,29)	c	a	$P_D[c,e]$ is updated from (14,31) to (16,29); $pred[c,e]$ is updated from c to a
7	a	d	b	(22,23)	(13,32)	(20,25)	d	a	
8	a	d	c	(15,30)	(13,32)	(24,21)	d	a	
9	a	d	e	(28,17)	(13,32)	(16,29)	d	a	
10	a	e	b	(27,18)	(29,16)	(20,25)	e	a	
11	a	e	c	(31,14)	(29,16)	(24,21)	e	a	
12	a	e	d	(17,28)	(29,16)	(32,13)	e	a	$P_D[e,d]$ is updated from (17,28) to (29,16); $pred[e,d]$ is updated from e to a
13	b	a	c	(24,21)	(20,25)	(26,19)	a	b	
14	b	a	d	(32,13)	(20,25)	(25,20)	a	a	
15	b	a	e	(16,29)	(20,25)	(18,27)	a	b	$P_D[a,e]$ is updated from (16,29) to (18,27); $pred[a,e]$ is updated from a to b
16	b	c	a	(21,24)	(20,25)	(25,20)	c	b	
17	b	c	d	(30,15)	(20,25)	(25,20)	c	a	
18	b	c	e	(16,29)	(20,25)	(18,27)	a	b	$P_D[c,e]$ is updated from (16,29) to (18,27); $pred[c,e]$ is updated from a to b
19	b	d	a	(13,32)	(22,23)	(25,20)	d	b	$P_D[d,a]$ is updated from (13,32) to (22,23); $pred[d,a]$ is updated from d to b
20	b	d	c	(15,30)	(22,23)	(26,19)	d	b	$P_D[d,c]$ is updated from (15,30) to (22,23); $pred[d,c]$ is updated from d to b
21	b	d	e	(28,17)	(22,23)	(18,27)	d	b	
22	b	e	a	(29,16)	(27,18)	(25,20)	e	b	
23	b	e	c	(31,14)	(27,18)	(26,19)	e	b	
24	b	e	d	(29,16)	(27,18)	(25,20)	a	a	
25	c	a	b	(20,25)	(24,21)	(20,25)	a	a	
26	c	a	d	(32,13)	(24,21)	(30,15)	a	c	
27	c	a	e	(18,27)	(24,21)	(18,27)	b	b	
28	c	b	a	(25,20)	(26,19)	(21,24)	b	c	
29	c	b	d	(25,20)	(26,19)	(30,15)	a	c	$P_D[b,d]$ is updated from (25,20) to (26,19); $pred[b,d]$ is updated from a to c
30	c	b	e	(18,27)	(26,19)	(18,27)	b	b	

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
31	*c*	*d*	*a*	(22,23)	(22,23)	(21,24)	*b*	*c*	
32	*c*	*d*	*b*	(22,23)	(22,23)	(20,25)	*d*	*a*	
33	*c*	*d*	*e*	(28,17)	(22,23)	(18,27)	*d*	*b*	
34	*c*	*e*	*a*	(29,16)	(31,14)	(21,24)	*e*	*c*	
35	*c*	*e*	*b*	(27,18)	(31,14)	(20,25)	*e*	*a*	
36	*c*	*e*	*d*	(29,16)	(31,14)	(30,15)	*a*	*c*	$P_D[e,d]$ is updated from (29,16) to (30,15); $pred[e,d]$ is updated from *a* to *c*
37	*d*	*a*	*b*	(20,25)	(32,13)	(22,23)	*a*	*d*	$P_D[a,b]$ is updated from (20,25) to (22,23); $pred[a,b]$ is updated from *a* to *d*
38	*d*	*a*	*c*	(24,21)	(32,13)	(22,23)	*a*	*b*	
39	*d*	*a*	*e*	(18,27)	(32,13)	(28,17)	*b*	*d*	$P_D[a,e]$ is updated from (18,27) to (28,17); $pred[a,e]$ is updated from *b* to *d*
40	*d*	*b*	*a*	(25,20)	(26,19)	(22,23)	*b*	*b*	
41	*d*	*b*	*c*	(26,19)	(26,19)	(22,23)	*b*	*b*	
42	*d*	*b*	*e*	(18,27)	(26,19)	(28,17)	*b*	*d*	$P_D[b,e]$ is updated from (18,27) to (26,19); $pred[b,e]$ is updated from *b* to *d*
43	*d*	*c*	*a*	(21,24)	(30,15)	(22,23)	*c*	*b*	$P_D[c,a]$ is updated from (21,24) to (22,23); $pred[c,a]$ is updated from *c* to *b*
44	*d*	*c*	*b*	(20,25)	(30,15)	(22,23)	*a*	*d*	$P_D[c,b]$ is updated from (20,25) to (22,23); $pred[c,b]$ is updated from *a* to *d*
45	*d*	*c*	*e*	(18,27)	(30,15)	(28,17)	*b*	*d*	$P_D[c,e]$ is updated from (18,27) to (28,17); $pred[c,e]$ is updated from *b* to *d*
46	*d*	*e*	*a*	(29,16)	(30,15)	(22,23)	*e*	*b*	
47	*d*	*e*	*b*	(27,18)	(30,15)	(22,23)	*e*	*d*	
48	*d*	*e*	*c*	(31,14)	(30,15)	(22,23)	*e*	*b*	
49	*e*	*a*	*b*	(22,23)	(28,17)	(27,18)	*d*	*e*	$P_D[a,b]$ is updated from (22,23) to (27,18); $pred[a,b]$ is updated from *d* to *e*
50	*e*	*a*	*c*	(24,21)	(28,17)	(31,14)	*a*	*e*	$P_D[a,c]$ is updated from (24,21) to (28,17); $pred[a,c]$ is updated from *a* to *e*
51	*e*	*a*	*d*	(32,13)	(28,17)	(30,15)	*a*	*c*	
52	*e*	*b*	*a*	(25,20)	(26,19)	(29,16)	*b*	*e*	$P_D[b,a]$ is updated from (25,20) to (26,19); $pred[b,a]$ is updated from *b* to *e*
53	*e*	*b*	*c*	(26,19)	(26,19)	(31,14)	*b*	*e*	
54	*e*	*b*	*d*	(26,19)	(26,19)	(30,15)	*c*	*c*	
55	*e*	*c*	*a*	(22,23)	(28,17)	(29,16)	*b*	*e*	$P_D[c,a]$ is updated from (22,23) to (28,17); $pred[c,a]$ is updated from *b* to *e*
56	*e*	*c*	*b*	(22,23)	(28,17)	(27,18)	*d*	*e*	$P_D[c,b]$ is updated from (22,23) to (27,18); $pred[c,b]$ is updated from *d* to *e*
57	*e*	*c*	*d*	(30,15)	(28,17)	(30,15)	*c*	*c*	
58	*e*	*d*	*a*	(22,23)	(28,17)	(29,16)	*b*	*e*	$P_D[d,a]$ is updated from (22,23) to (28,17); $pred[d,a]$ is updated from *b* to *e*
59	*e*	*d*	*b*	(22,23)	(28,17)	(27,18)	*d*	*e*	$P_D[d,b]$ is updated from (22,23) to (27,18); $pred[d,b]$ is updated from *d* to *e*
60	*e*	*d*	*c*	(22,23)	(28,17)	(31,14)	*b*	*e*	$P_D[d,c]$ is updated from (22,23) to (28,17); $pred[d,c]$ is updated from *b* to *e*

3.10. Example 10

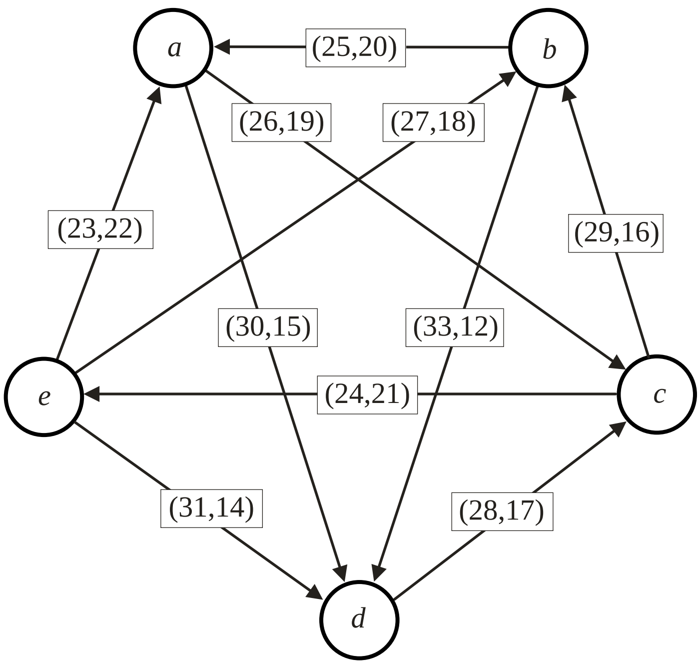
Example 10:

5	voters	$a \succ_v c \succ_v b \succ_v e \succ_v d$
5	voters	$a \succ_v d \succ_v e \succ_v c \succ_v b$
8	voters	$b \succ_v e \succ_v d \succ_v a \succ_v c$
3	voters	$c \succ_v a \succ_v b \succ_v e \succ_v d$
7	voters	$c \succ_v a \succ_v e \succ_v b \succ_v d$
2	voters	$c \succ_v b \succ_v a \succ_v d \succ_v e$
7	voters	$d \succ_v c \succ_v e \succ_v b \succ_v a$
8	voters	$e \succ_v b \succ_v a \succ_v d \succ_v c$

The pairwise matrix N looks as follows:

	$N[*,a]$	$N[*,b]$	$N[*,c]$	$N[*,d]$	$N[*,e]$
$N[a,*]$	---	20	26	30	22
$N[b,*]$	25	---	16	33	18
$N[c,*]$	19	29	---	17	24
$N[d,*]$	15	12	28	---	14
$N[e,*]$	23	27	21	31	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*	... to *e*
from *a* ...	---	*a*, (30,15), *d*, (28,17), *c*, (29,16), *b*	*a*, (30,15), *d*, (28,17), *c*	*a*, (30,15), *d*	*a*, (30,15), *d*, (28,17), *c*, (24,21), *e*
from *b* ...	*b*, (25,20), *a*	---	*b*, (33,12), *d*, (28,17), *c*	*b*, (33,12), *d*	*b*, (33,12), *d*, (28,17), *c*, (24,21), *e*
from *c* ...	*c*, (29,16), *b*, (25,20), *a*	*c*, (29,16), *b*	---	*c*, (29,16), *b*, (33,12), *d*	*c*, (24,21), *e*
from *d* ...	*d*, (28,17), *c*, (29,16), *b*, (25,20), *a*	*d*, (28,17), *c*, (29,16), *b*	*d*, (28,17), *c*	---	*d*, (28,17), *c*, (24,21), *e*
from *e* ...	*e*, (31,14), *d*, (28,17), *c*, (29,16), *b*, (25,20), *a*	*e*, (31,14), *d*, (28,17), *c*, (29,16), *b*	*e*, (31,14), *d*, (28,17), *c*	*e*, (31,14), *d*	---

We get $\mathcal{O} = \{ab, ac, ad, bd, cb, cd, ea, eb, ec, ed\}$ and $\mathcal{S} = \{e\}$.

Suppose, the strongest paths are calculated with the Floyd algorithm, as defined in section 2.3. Then the following table documents the 60 steps of the Floyd algorithm.

We start with

- $P_D[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(16,29)	(25,20)	(26,19)	b	a	$P_D[b,c]$ is updated from (16,29) to (25,20); $pred[b,c]$ is updated from b to a
2	a	b	d	(33,12)	(25,20)	(30,15)	b	a	
3	a	b	e	(18,27)	(25,20)	(22,23)	b	a	$P_D[b,e]$ is updated from (18,27) to (22,23); $pred[b,e]$ is updated from b to a
4	a	c	b	(29,16)	(19,26)	(20,25)	c	a	
5	a	c	d	(17,28)	(19,26)	(30,15)	c	a	$P_D[c,d]$ is updated from (17,28) to (19,26); $pred[c,d]$ is updated from c to a
6	a	c	e	(24,21)	(19,26)	(22,23)	c	a	
7	a	d	b	(12,33)	(15,30)	(20,25)	d	a	$P_D[d,b]$ is updated from (12,33) to (15,30); $pred[d,b]$ is updated from d to a
8	a	d	c	(28,17)	(15,30)	(26,19)	d	a	
9	a	d	e	(14,31)	(15,30)	(22,23)	d	a	$P_D[d,e]$ is updated from (14,31) to (15,30); $pred[d,e]$ is updated from d to a
10	a	e	b	(27,18)	(23,22)	(20,25)	e	a	
11	a	e	c	(21,24)	(23,22)	(26,19)	e	a	$P_D[e,c]$ is updated from (21,24) to (23,22); $pred[e,c]$ is updated from e to a
12	a	e	d	(31,14)	(23,22)	(30,15)	e	a	
13	b	a	c	(26,19)	(20,25)	(25,20)	a	a	
14	b	a	d	(30,15)	(20,25)	(33,12)	a	b	
15	b	a	e	(22,23)	(20,25)	(22,23)	a	a	
16	b	c	a	(19,26)	(29,16)	(25,20)	c	b	$P_D[c,a]$ is updated from (19,26) to (25,20); $pred[c,a]$ is updated from c to b
17	b	c	d	(19,26)	(29,16)	(33,12)	a	b	$P_D[c,d]$ is updated from (19,26) to (29,16); $pred[c,d]$ is updated from a to b
18	b	c	e	(24,21)	(29,16)	(22,23)	c	a	
19	b	d	a	(15,30)	(15,30)	(25,20)	d	b	
20	b	d	c	(28,17)	(15,30)	(25,20)	d	a	
21	b	d	e	(15,30)	(15,30)	(22,23)	a	a	
22	b	e	a	(23,22)	(27,18)	(25,20)	e	b	$P_D[e,a]$ is updated from (23,22) to (25,20); $pred[e,a]$ is updated from e to b
23	b	e	c	(23,22)	(27,18)	(25,20)	a	a	$P_D[e,c]$ is updated from (23,22) to (25,20)
24	b	e	d	(31,14)	(27,18)	(33,12)	e	b	
25	c	a	b	(20,25)	(26,19)	(29,16)	a	c	$P_D[a,b]$ is updated from (20,25) to (26,19); $pred[a,b]$ is updated from a to c
26	c	a	d	(30,15)	(26,19)	(29,16)	a	b	
27	c	a	e	(22,23)	(26,19)	(24,21)	a	c	$P_D[a,e]$ is updated from (22,23) to (24,21); $pred[a,e]$ is updated from a to c
28	c	b	a	(25,20)	(25,20)	(25,20)	b	b	
29	c	b	d	(33,12)	(25,20)	(29,16)	b	b	
30	c	b	e	(22,23)	(25,20)	(24,21)	a	c	$P_D[b,e]$ is updated from (22,23) to (24,21); $pred[b,e]$ is updated from a to c

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
31	c	d	a	(15,30)	(28,17)	(25,20)	d	b	$P_D[d,a]$ is updated from (15,30) to (25,20); $pred[d,a]$ is updated from d to b
32	c	d	b	(15,30)	(28,17)	(29,16)	a	c	$P_D[d,b]$ is updated from (15,30) to (28,17); $pred[d,b]$ is updated from a to c
33	c	d	e	(15,30)	(28,17)	(24,21)	a	c	$P_D[d,e]$ is updated from (15,30) to (24,21); $pred[d,e]$ is updated from a to c
34	c	e	a	(25,20)	(25,20)	(25,20)	b	b	
35	c	e	b	(27,18)	(25,20)	(29,16)	e	c	
36	c	e	d	(31,14)	(25,20)	(29,16)	e	b	
37	d	a	b	(26,19)	(30,15)	(28,17)	c	c	$P_D[a,b]$ is updated from (26,19) to (28,17)
38	d	a	c	(26,19)	(30,15)	(28,17)	a	d	$P_D[a,c]$ is updated from (26,19) to (28,17); $pred[a,c]$ is updated from a to d
39	d	a	e	(24,21)	(30,15)	(24,21)	c	c	
40	d	b	a	(25,20)	(33,12)	(25,20)	b	b	
41	d	b	c	(25,20)	(33,12)	(28,17)	a	d	$P_D[b,c]$ is updated from (25,20) to (28,17); $pred[b,c]$ is updated from a to d
42	d	b	e	(24,21)	(33,12)	(24,21)	c	c	
43	d	c	a	(25,20)	(29,16)	(25,20)	b	b	
44	d	c	b	(29,16)	(29,16)	(28,17)	c	c	
45	d	c	e	(24,21)	(29,16)	(24,21)	c	c	
46	d	e	a	(25,20)	(31,14)	(25,20)	b	b	
47	d	e	b	(27,18)	(31,14)	(28,17)	e	c	$P_D[e,b]$ is updated from (27,18) to (28,17); $pred[e,b]$ is updated from e to c
48	d	e	c	(25,20)	(31,14)	(28,17)	a	d	$P_D[e,c]$ is updated from (25,20) to (28,17); $pred[e,c]$ is updated from a to d
49	e	a	b	(28,17)	(24,21)	(28,17)	c	c	
50	e	a	c	(28,17)	(24,21)	(28,17)	d	d	
51	e	a	d	(30,15)	(24,21)	(31,14)	a	e	
52	e	b	a	(25,20)	(24,21)	(25,20)	b	b	
53	e	b	c	(28,17)	(24,21)	(28,17)	d	d	
54	e	b	d	(33,12)	(24,21)	(31,14)	b	e	
55	e	c	a	(25,20)	(24,21)	(25,20)	b	b	
56	e	c	b	(29,16)	(24,21)	(28,17)	c	c	
57	e	c	d	(29,16)	(24,21)	(31,14)	b	e	
58	e	d	a	(25,20)	(24,21)	(25,20)	b	b	
59	e	d	b	(28,17)	(24,21)	(28,17)	c	c	
60	e	d	c	(28,17)	(24,21)	(28,17)	d	d	

4. Analysis of the Schulze Method

4.1. Transitivity

In this section, we will prove that the binary relation $\mathcal{O}$, as defined in (2.2.1), is *transitive*. This means: If $ab \in \mathcal{O}$ and $bc \in \mathcal{O}$, then $ac \in \mathcal{O}$. This guarantees that the set $\mathcal{S}$ of potential winners, as defined in (2.2.2), is non-empty. When we interpret the Schulze method as a method to find a set $\mathcal{S}$ of potential winners, rather than a method to generate a binary relation $\mathcal{O}$, then the proof of the transitivity of $\mathcal{O}$ is an essential part of the proof that the Schulze method is well defined.

Definition:

An election method satisfies *transitivity* if the following holds for all $a,b,c \in A$:

Suppose:

(4.1.1) $ab \in \mathcal{O}$.

(4.1.2) $bc \in \mathcal{O}$.

Then:

(4.1.3) $ac \in \mathcal{O}$.

Claim:

The binary relation $\mathcal{O}$, as defined in (2.2.1), is transitive.

Proof:

With (4.1.1), we get

(4.1.4) $P_D[a,b] \succ_D P_D[b,a]$.

With (4.1.2), we get

(4.1.5) $P_D[b,c] \succ_D P_D[c,b]$.

With (2.2.5), we get

(4.1.6) $\min_D \{ P_D[a,b], P_D[b,c] \} \precsim_D P_D[a,c]$.

(4.1.7) $\min_D \{ P_D[b,c], P_D[c,a] \} \precsim_D P_D[b,a]$.

(4.1.8) $\min_D \{ P_D[c,a], P_D[a,b] \} \precsim_D P_D[c,b]$.

Case 1: Suppose

(4.1.9a) $P_D[a,b] \succsim_D P_D[b,c]$.

Combining (4.1.5) and (4.1.9a) gives

(4.1.10a) $P_D[a,b] \succ_D P_D[c,b]$.

Combining (4.1.8) and (4.1.10a) gives

(4.1.11a) $P_D[c,a] \precsim_D P_D[c,b]$.

Combining (4.1.6) and (4.1.9a) gives

(4.1.12a) $P_D[b,c] \precsim_D P_D[a,c]$.

Combining (4.1.11a), (4.1.5), and (4.1.12a) gives

(4.1.13a) $P_D[c,a] \precsim_D P_D[c,b] \prec_D P_D[b,c] \precsim_D P_D[a,c]$.

With (4.1.13a), we get (4.1.3).

Case 2: Suppose

(4.1.9b) $P_D[a,b] \prec_D P_D[b,c]$.

Combining (4.1.4) and (4.1.9b) gives

(4.1.10b) $P_D[b,a] \prec_D P_D[b,c]$.

Combining (4.1.7) and (4.1.10b) gives

(4.1.11b) $P_D[c,a] \precsim_D P_D[b,a]$.

Combining (4.1.6) and (4.1.9b) gives

(4.1.12b) $P_D[a,b] \precsim_D P_D[a,c]$.

Combining (4.1.11b), (4.1.4), and (4.1.12b) gives

(4.1.13b) $P_D[c,a] \precsim_D P_D[b,a] \prec_D P_D[a,b] \precsim_D P_D[a,c]$.

With (4.1.13b), we get (4.1.3). □

The proof, that the Schulze method is transitive, has first been published by Schulze (1998).

The following corollary says that the set $\mathcal{S}$ of potential winners, as defined in (2.2.2), is non-empty.

Corollary (4.1.14):

For the Schulze method, as defined in section 2.2, we get

(4.1.14) $\forall\ b \notin \mathcal{S}\ \exists\ a \in \mathcal{S}\text{: } ab \in \mathcal{O}$.

Proof of corollary (4.1.14):

As $b \notin \mathcal{S}$, there must be a $c(1) \in A$ with $c(1),b \in \mathcal{O}$.

If $c(1) \in \mathcal{S}$, then the corollary is proven. If $c(1) \notin \mathcal{S}$, then there must be a $c(2) \in A$ with $c(2),c(1) \in \mathcal{O}$. With the asymmetry and the transitivity of $\mathcal{O}$, we get $c(2),b \in \mathcal{O}$ and $c(2) \notin \{b, c(1)\}$.

We now proceed as follows: If $c(i) \in \mathcal{S}$, then the corollary is proven. If $c(i) \notin \mathcal{S}$, then there must be a $c(i+1) \in A$ with $c(i+1),c(i) \in \mathcal{O}$. With the asymmetry and the transitivity of $\mathcal{O}$, we get $c(i+1),b \in \mathcal{O}$ and $c(i+1) \notin \{b, c(1), ..., c(i)\}$.

We proceed until $c(i) \in \mathcal{S}$ for some $i \in \mathbb{N}$. Such an $i \in \mathbb{N}$ exists because A is finite. □

The following corollary says that alternative $a \in A$ is the unique winner if and only if alternative a disqualifies every other alternative $b \in A \setminus \{a\}$.

Corollary (4.1.15):

For the Schulze method, as defined in section 2.2, we get

(4.1.15) $\mathcal{S} = \{a\} \Leftrightarrow ab \in \mathcal{O}\ \forall\ b \in A \setminus \{a\}$.

Proof of corollary (4.1.15):

$\Leftarrow$ If $ab \in \mathcal{O}\ \forall\ b \in A \setminus \{a\}$, then $a \in A$ disqualifies every $b \in A \setminus \{a\}$ according to (2.2.2). Therefore, we get $\mathcal{S} = \{a\}$.

$\Rightarrow$ With (4.1.14) and $\mathcal{S} = \{a\}$, we get

(4.1.16) $\forall\ b \notin \mathcal{S}\text{: } ab \in \mathcal{O}$.

With $\mathcal{S} = \{a\}$, we get

(4.1.17) $b \notin \mathcal{S} \Leftrightarrow b \in A \setminus \{a\}$.

With (4.1.16) and (4.1.17), we get

(4.1.18) $\forall\ b \in A \setminus \{a\}\text{: } ab \in \mathcal{O}$. □

In example 2 (section 3.2), we have $ba \notin \mathcal{O}$ and $ac \notin \mathcal{O}$ and $bc \in \mathcal{O}$. This shows that the Schulze relation, as defined in (2.2.1), is not necessarily negatively transitive.

4.2. Resolvability

Resolvability basically says that usually there is a unique winner $\mathcal{S} = \{a\}$. There are two different versions of the resolvability criterion. We will prove that the Schulze method, as defined in section 2.2, satisfies both.

4.2.1. Formulation #1

Definition:

An election method satisfies the first version of the *resolvability criterion* if (for every given number of alternatives) the proportion of profiles without a unique winner tends to zero as the number of voters in the profile tends to infinity.

Claim:

If $\succ_D$ satisfies (2.1.1), then the Schulze method, as defined in section 2.2, satisfies the first version of the resolvability criterion.

Proof (overview):

Suppose $(x_1,x_2),(y_1,y_2) \in \mathbb{N}_0 \times \mathbb{N}_0$. Then, according to (2.1.1), there is a $v_1 \in \mathbb{N}_0$ such that for all $w_1 \in \mathbb{N}_0$:

1. $w_1 < v_1 \Rightarrow (x_1,x_2) \succ_D (w_1,y_2)$.

2. $w_1 > v_1 \Rightarrow (x_1,x_2) \prec_D (w_1,y_2)$.

When the number of voters tends to infinity (i.e. when x_1, x_2, y_1, and y_2 become large), then the proportion of profiles, where the condition "$y_1 = v_1$" happens to be satisfied, tends to zero. Therefore, when the number of voters tends to infinity, then the proportion of profiles, where two links *ef* and *gh* happen to have equivalent strengths $(N[e,f],N[f,e]) \approx_D (N[g,h],N[h,g])$, tends to zero.

Therefore, we will prove that, unless there are links *ef* and *gh* of equivalent strengths, there is always a unique winner. We will prove this by showing that, when we simultaneously presume (a) that there is more than one potential winner and (b) that there are no links *ef* and *gh* of equivalent strengths, then we necessarily get to a contradiction.

Proof (details):

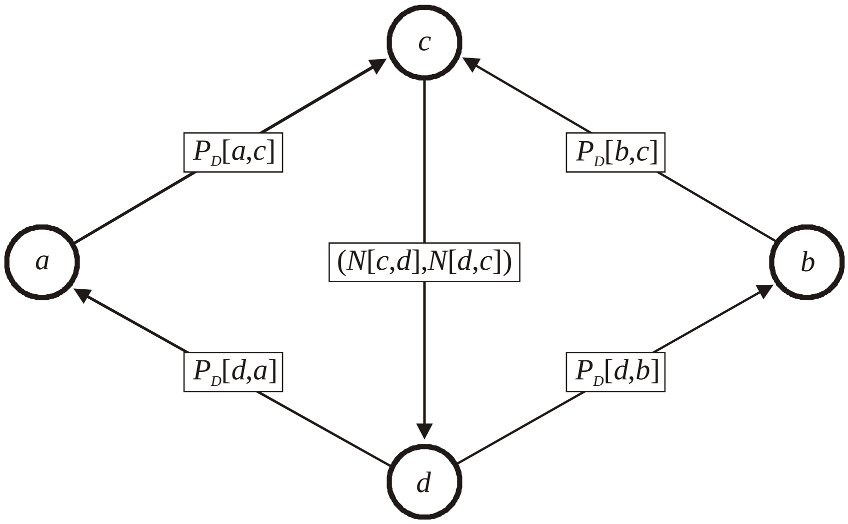
Suppose that there is more than one potential winner. Suppose alternative $a \in A$ and alternative $b \in A$ are potential winners. Then

(4.2.1.1) $\forall\ i \in A \setminus \{a\}: P_D[a,i] \succsim_D P_D[i,a]$.

(4.2.1.2) $\forall\ j \in A \setminus \{b\}: P_D[b,j] \succsim_D P_D[j,b]$.

(4.2.1.3) $P_D[a,b] \approx_D P_D[b,a]$.

Suppose there are no links *ef* and *gh* of equivalent strengths $(N[e,f],N[f,e]) \approx_D (N[g,h],N[h,g])$. Then $P_D[a,b] \approx_D P_D[b,a]$ means that the weakest link in the strongest path from alternative *a* to alternative *b* and the weakest link in the strongest path from alternative *b* to alternative *a* must be the same link, say *cd*. Therefore, the strongest paths have the following structure:



As *cd* is the weakest link in the strongest path from alternative *a* to alternative *b*, we get

(4.2.1.4) $\qquad P_D[a,d] \approx_D P_D[a,b]$.

(4.2.1.5) $\qquad P_D[d,b] \succ_D P_D[a,b]$.

As *cd* is the weakest link in the strongest path from alternative *b* to alternative *a*, we get

(4.2.1.6) $\qquad P_D[b,d] \approx_D P_D[b,a]$.

(4.2.1.7) $\qquad P_D[d,a] \succ_D P_D[b,a]$.

With (4.2.1.7), (4.2.1.3), and (4.2.1.4), we get

(4.2.1.8) $\qquad P_D[d,a] \succ_D P_D[b,a] \approx_D P_D[a,b] \approx_D P_D[a,d]$.

But (4.2.1.8) contradicts (4.2.1.1).

Similarly, with (4.2.1.5), (4.2.1.3), and (4.2.1.6), we get

(4.2.1.9) $\qquad P_D[d,b] \succ_D P_D[a,b] \approx_D P_D[b,a] \approx_D P_D[b,d]$.

But (4.2.1.9) contradicts (4.2.1.2). □

4.2.2. Formulation #2

The second version of the *resolvability criterion* says that, when there is more than one potential winner, then, for every alternative $a \in \mathcal{S}$, it is sufficient to add a single ballot w so that alternative a becomes the unique winner.

Definition:

An election method satisfies the second version of the *resolvability criterion* if the following holds:

$\forall\ a \in \mathcal{S}^{\text{old}}$: It is possible to construct a strict weak order w with the following two properties:

(4.2.2.1) $\quad \forall\ f \in A \setminus \{a\}: a \succ_w f.$

(4.2.2.2) $\quad \mathcal{S}^{\text{new}} = \{a\}$ for $V^{\text{new}} := V^{\text{old}} + \{w\}$.

Claim:

If $\succ_D$ satisfies (2.1.1), then the Schulze method, as defined in section 2.2, satisfies the second version of the resolvability criterion.

Proof:

Suppose $a \in \mathcal{S}^{\text{old}}$. Then we get

(4.2.2.3) $\quad \forall\ b \in A \setminus \{a\}: P_D^{\text{old}}[a,b] \succsim_D P_D^{\text{old}}[b,a].$

Suppose $pred^{\text{old}}[x,y]$ is the predecessor of alternative y in the strongest path from alternative $x \in A$ to alternative $y \in A \setminus \{x\}$, as calculated in section 2.3.

Suppose the strict weak order w is chosen as follows:

(4.2.2.4) $\quad \forall\ f \in A \setminus \{a\}: pred^{\text{old}}[a,f] \succ_w f.$

(4.2.2.5) $\quad \forall\ e,f \in A \setminus \{a\}: (\ P_D^{\text{old}}[e,a] \succ_D P_D^{\text{old}}[f,a] \Rightarrow e \succ_w f\ ).$

With (4.2.2.4), we get (4.2.2.1).

We will prove the following three claims:

Claim #1: It is not possible that (4.2.2.4) requires $e \succ_w f$ and that simultaneously (4.2.2.5) requires $f \succ_w e$.

Claim #2: $\forall\ g \in A \setminus \{a\}: P_D^{\text{new}}[a,g] \succ_D P_D^{\text{old}}[a,g].$

Claim #3: $\forall\ g \in A \setminus \{a\}: P_D^{\text{new}}[g,a] \prec_D P_D^{\text{old}}[a,g].$

With claim #2 and claim #3, we get

$P_D^{\text{new}}[a,g] \succ_D P_D^{\text{new}}[g,a]$ for all $g \in A \setminus \{a\}$

so that $ag \in \mathcal{O}^{\text{new}}$ for all $g \in A \setminus \{a\}$

so that $\mathcal{S}^{\text{new}} = \{a\}$.

Proof of claim #1:

Suppose $e,f \in A \setminus \{a\}$. With (2.2.3), we get

(4.2.2.6) $P_D^{\text{old}}[e,f] \approx_D (N^{\text{old}}[e,f],N^{\text{old}}[f,e])$.

With (2.2.5), we get

(4.2.2.7) $\min_D \{ P_D^{\text{old}}[e,f], P_D^{\text{old}}[f,a] \} \precsim_D P_D^{\text{old}}[e,a]$.

With (4.2.2.3), we get

(4.2.2.8) $P_D^{\text{old}}[a,f] \succsim_D P_D^{\text{old}}[f,a]$.

Suppose (4.2.2.4) requires $e \succ_w f$. Then $e = pred^{\text{old}}[a,f]$. Therefore, the link ef was in the strongest path from alternative a to alternative f. Thus, we get

(4.2.2.9) $P_D^{\text{old}}[a,f] \precsim_D (N^{\text{old}}[e,f],N^{\text{old}}[f,e])$.

Suppose (4.2.2.5) requires $f \succ_w e$. Then

(4.2.2.10) $P_D^{\text{old}}[f,a] \succ_D P_D^{\text{old}}[e,a]$.

With (4.2.2.6), (4.2.2.9), (4.2.2.8), and (4.2.2.10), we get

(4.2.2.11) $P_D^{\text{old}}[e,f] \approx_D (N^{\text{old}}[e,f],N^{\text{old}}[f,e]) \succsim_D P_D^{\text{old}}[a,f] \succsim_D P_D^{\text{old}}[f,a] \succ_D P_D^{\text{old}}[e,a]$.

But (4.2.2.10) and (4.2.2.11) together contradict (4.2.2.7).

Proof of claim #2:

With (2.1.1) and (4.2.2.4), we get: The strength of each link of the strongest paths from alternative a to each other alternative $g \in A \setminus \{a\}$ is increased. Therefore

(4.2.2.12) $\forall\, g \in A \setminus \{a\}: P_D^{\text{new}}[a,g] \succ_D P_D^{\text{old}}[a,g]$.

Proof of claim #3:

Suppose $g \in A \setminus \{a\}$. Suppose

(4.2.2.13) $\Upsilon(g) := (\{a\} \cup \{ h \in A \setminus \{a\} \mid P^{\text{old}}_{D}[h,a] \succ_D P^{\text{old}}_{D}[a,g] \})$.

With (4.2.2.3) and (4.2.2.13), we get

(4.2.2.14) $g \notin \Upsilon(g)$ and $a \in \Upsilon(g)$

and, therefore, $\emptyset \neq \Upsilon(g) \subsetneq A$. Furthermore, we get

(4.2.2.15) $\forall\, i \notin \Upsilon(g)\ \forall\, j \in \Upsilon(g)$: $(N^{\text{old}}[i,j],N^{\text{old}}[j,i]) \precsim_D P^{\text{old}}_{D}[a,g]$.

Otherwise, there was a path from alternative i to alternative a via alternative j with a strength of more than $P^{\text{old}}_{D}[a,g]$. But (as $i \notin \Upsilon(g)$) this would contradict the definition of $\Upsilon(g)$.

With (4.2.2.5), (4.2.2.1), and (4.2.2.13), we get

(4.2.2.16) $\forall\, i \notin \Upsilon(g)\ \forall\, j \in \Upsilon(g)$: $j \succ_w i$.

With (2.1.1) and (4.2.2.16), we get

(4.2.2.17) $\forall\, i \notin \Upsilon(g)\ \forall\, j \in \Upsilon(g)$: $(N^{\text{new}}[i,j],N^{\text{new}}[j,i]) \prec_D (N^{\text{old}}[i,j],N^{\text{old}}[j,i])$.

With (4.2.2.15) and (4.2.2.17), we get

(4.2.2.18) $\forall\, i \notin \Upsilon(g)\ \forall\, j \in \Upsilon(g)$: $(N^{\text{new}}[i,j],N^{\text{new}}[j,i]) \prec_D P^{\text{old}}_{D}[a,g]$.

With (4.2.2.14) and (4.2.2.18), we get

(4.2.2.19) $P^{\text{new}}_{D}[g,a] \prec_D P^{\text{old}}_{D}[a,g]$. □

The proof in section 4.2.2 has first been published by Schulze (2011). It immediately attracted attention, because it doesn’t only prove that there is a tie-breaking ballot *w*, it also shows how this tie-breaking ballot *w* can be calculated in a polynomial runtime. Parkes and Xia (2012) pointed to the fact that this proof can also be interpreted as saying that it is possible to calculate a voting strategy in a polynomial runtime. This observation by Parkes and Xia has been extended by Gaspers (2012), Menton (2013a, 2013b), J. Müller (2013), Reisch (2014), Schend (2015), and Hemaspaandra (2016). Surveys, that are including the Schulze method, on the complexity of calculating a voting strategy have been written by Durand (2015), Baumeister and Rothe (2016), Conitzer and Walsh (2016), and Faliszewski and Rothe (2016).

4.3. Pareto

The *Pareto criterion* says that the election method must respect unanimous opinions. There are two different versions of the Pareto criterion. The first version addresses situations with “ $a \succ_v b$ for all $v \in V$ ”, while the second version addresses situations with “ $a \succsim_v b$ for all $v \in V$ ” (for some pair of alternatives $a,b \in A$). The first version says that, when every voter strictly prefers alternative a to alternative b (i.e. $a \succ_v b$ for all $v \in V$), then alternative a must perform better than alternative b. The second version says that, when no voter strictly prefers alternative b to alternative a (i.e. $a \succsim_v b$ for all $v \in V$), then alternative b must not perform better than alternative a. We will prove that the Schulze method, as defined in section 2.2, satisfies both versions of the Pareto criterion.

4.3.1. Formulation #1

Definition:

An election method satisfies the first version of the *Pareto criterion* if the following holds:

Suppose:

(4.3.1.1) $\forall\ v \in V$: $a \succ_v b$.

Then:

(4.3.1.2) $ab \in \mathcal{O}$.

(4.3.1.3) $b \notin \mathcal{S}$.

Claim:

If $\succ_D$ satisfies (2.1.1), then the Schulze method, as defined in section 2.2, satisfies the first version of the Pareto criterion.

Proof:

With (2.1.1) and (4.3.1.1), we get

(4.3.1.4) $\forall\ e,f \in A$: $(N[a,b],N[b,a]) \succsim_D (N[e,f],N[f,e])$.

(4.3.1.5) $[\ (N[a,b],N[b,a]) \approx_D (N[e,f],N[f,e])\ ] \Leftrightarrow [\ \forall\ v \in V\text{: } e \succ_v f\ ]$.

With (2.2.4), we get: $ab \in \mathcal{O}$, unless the link ab is in a directed cycle that consists of links of which each is at least as strong as the link ab.

However, as we presumed that the individual ballots $\succ_v$ are strict weak orders, it is not possible that there is a directed cycle of unanimous opinions. Therefore, it is not possible that the link ab is in a directed cycle that consists of links of which each is at least as strong as the link ab. Therefore, with (2.2.4), (4.3.1.4), and (4.3.1.5), we get (4.3.1.2). With (4.3.1.2), we get (4.3.1.3). □

4.3.2. Formulation #2

Definition:

An election method satisfies the second version of the *Pareto criterion* if the following holds:

Suppose:

(4.3.2.1) $\forall\, v \in V$: $a \succsim_v b$.

Then:

(4.3.2.2) $ba \notin \mathcal{O}$.

(4.3.2.3) $\forall\, f \in A \setminus \{a,b\}$: $bf \in \mathcal{O} \Rightarrow af \in \mathcal{O}$.

(4.3.2.4) $\forall\, f \in A \setminus \{a,b\}$: $fa \in \mathcal{O} \Rightarrow fb \in \mathcal{O}$.

(4.3.2.5) $b \in \mathcal{S} \Rightarrow a \in \mathcal{S}$.

Claim:

If $\succ_D$ satisfies (2.1.1), then the Schulze method, as defined in section 2.2, satisfies the second version of the Pareto criterion.

Proof:

With (4.3.2.1), we get

(4.3.2.6) $\forall\, e \in A \setminus \{a,b\}$: $N[a,e] \geq N[b,e]$.

With (4.3.2.1), we get

(4.3.2.7) $\forall\, e \in A \setminus \{a,b\}$: $N[e,b] \geq N[e,a]$.

With (2.1.1), (4.3.2.6), and (4.3.2.7), we get

(4.3.2.8) $\forall\, e \in A \setminus \{a,b\}$: $(N[a,e],N[e,a]) \succsim_D (N[b,e],N[e,b])$.

With (2.1.1), (4.3.2.6), and (4.3.2.7), we get

(4.3.2.9) $\forall\, e \in A \setminus \{a,b\}$: $(N[e,b],N[b,e]) \succsim_D (N[e,a],N[a,e])$.

Suppose $c(1),\ldots,c(n) \in A$ is the strongest path from alternative b to alternative a. With (4.3.2.8) and (4.3.2.9), we get: $a,c(2),\ldots,c(n-1),b$ is a path from alternative a to alternative b with at least the same strength. Therefore

(4.3.2.10) $P_D[a,b] \succsim_D P_D[b,a]$.

With (4.3.2.10), we get (4.3.2.2).

Suppose $c(1),\ldots,c(n) \in A$ is the strongest path from alternative b to alternative $f \in A \setminus \{a,b\}$. With (4.3.2.8), we get: $a,c(m+1),\ldots,c(n)$, where $c(m)$

is the last occurrence of an alternative of the set $\{a,b\}$, is a path from alternative a to alternative f with at least the same strength. Therefore

(4.3.2.11) $\forall\, f \in A \setminus \{a,b\}$: $P_D[a,f] \gtrsim_D P_D[b,f]$.

Suppose $c(1),...,c(n) \in A$ is the strongest path from alternative $f \in A \setminus \{a,b\}$ to alternative a. With (4.3.2.9), we get: $c(1),...,c(m-1),b$, where $c(m)$ is the first occurrence of an alternative of the set $\{a,b\}$, is a path from alternative f to alternative b with at least the same strength. Therefore

(4.3.2.12) $\forall\, f \in A \setminus \{a,b\}$: $P_D[f,b] \gtrsim_D P_D[f,a]$.

<u>Part 1:</u> Suppose $f \in A \setminus \{a,b\}$. Suppose

(4.3.2.13a) $bf \in \mathcal{O}$.

With (4.3.2.13a), we get

(4.3.2.14a) $P_D[b,f] \succ_D P_D[f,b]$.

With (4.3.2.11), (4.3.2.14a), and (4.3.2.12), we get

(4.3.2.15a) $P_D[a,f] \gtrsim_D P_D[b,f] \succ_D P_D[f,b] \gtrsim_D P_D[f,a]$.

With (4.3.2.15a), we get (4.3.2.3).

<u>Part 2:</u> Suppose $f \in A \setminus \{a,b\}$. Suppose

(4.3.2.13b) $fa \in \mathcal{O}$.

With (4.3.2.13b), we get

(4.3.2.14b) $P_D[f,a] \succ_D P_D[a,f]$.

With (4.3.2.12), (4.3.2.14b), and (4.3.2.11), we get

(4.3.2.15b) $P_D[f,b] \gtrsim_D P_D[f,a] \succ_D P_D[a,f] \gtrsim_D P_D[b,f]$.

With (4.3.2.15b), we get (4.3.2.4).

<u>Part 3:</u> Suppose

(4.3.2.13c) $b \in \mathcal{S}$.

With (4.3.2.13c), we get

(4.3.2.14c) $\forall\, f \in A \setminus \{b\}$: $fb \notin \mathcal{O}$.

With (4.3.2.4) and (4.3.2.14c), we get

(4.3.2.15c) $\forall\, f \in A \setminus \{a,b\}$: $fa \notin \mathcal{O}$.

With (4.3.2.2) and (4.3.2.15c), we get

(4.3.2.16c) $\forall\, f \in A \setminus \{a\}$: $fa \notin \mathcal{O}$.

With (4.3.2.16c), we get (4.3.2.5). □

4.4. Reversal Symmetry

Reversal symmetry as a criterion for single-winner election methods has been proposed by Saari (1994). This criterion says that, when $\succ_v$ is reversed for all $v \in V$, then also the result of the elections must be reversed; see (4.4.2). $\mathcal{S}^{\text{old}}$ must not be a strict subset of $\mathcal{S}^{\text{new}}$; $\mathcal{S}^{\text{new}}$ must not be a strict subset of $\mathcal{S}^{\text{old}}$; see (4.4.3). It should not be possible that the same alternatives are elected in the original situation and in the reversed situation, unless all alternatives are tied; see (4.4.4).

Basic idea of this criterion is that, when there is a vote on the best alternatives and then there is a vote on the worst alternatives and when in both cases the same alternatives are chosen, then this questions the logic of the underlying heuristic of the used election method.

<u>Definition:</u>

An election method satisfies *reversal symmetry* if the following holds:

Suppose:

(4.4.1) $\forall\ e,f \in A\ \forall\ v \in V$: $e \succ_v^{\text{old}} f \Leftrightarrow f \succ_v^{\text{new}} e$.

Then:

(4.4.2) $\forall\ a,b \in A$: $ab \in \mathcal{O}^{\text{old}} \Leftrightarrow ba \in \mathcal{O}^{\text{new}}$.

(4.4.3) $(\ \exists\ i \in A$: $i \in \mathcal{S}^{\text{old}} \wedge i \notin \mathcal{S}^{\text{new}}\ ) \Leftrightarrow$
$(\ \exists\ j \in A$: $j \notin \mathcal{S}^{\text{old}} \wedge j \in \mathcal{S}^{\text{new}}\ )$.

(4.4.4) $\mathcal{S}^{\text{old}} = \mathcal{S}^{\text{new}} \Leftrightarrow \mathcal{S}^{\text{old}} = A$.

<u>Claim:</u>

The Schulze method, as defined in section 2.2, satisfies reversal symmetry.

<u>Proof:</u>

With (4.4.1), we get

(4.4.5) $\forall\ e,f \in A$: $N^{old}[e,f] = N^{new}[f,e]$.

With (4.4.5), we get

(4.4.6) $\forall\ e,f \in A$: $(N^{old}[e,f],N^{old}[f,e]) \approx_D (N^{new}[f,e],N^{new}[e,f])$.

With (4.4.6), we get: When $c(1),...,c(n) \in A$ was a path from alternative $g \in A$ to alternative $h \in A \setminus \{g\}$, then $c(n),...,c(1)$ is a path from alternative h to alternative g with the same strength. Therefore

(4.4.7) $\forall\ g,h \in A$: $P_D^{old}[g,h] \approx_D P_D^{new}[h,g]$.

With (4.4.7), we get (4.4.2).

<u>Part 1:</u>

Suppose $\exists\ i \in A$: $i \in \mathcal{S}^{old}$ and $i \notin \mathcal{S}^{new}$. With $i \notin \mathcal{S}^{new}$ and (4.1.14), we get that there is a $j \in \mathcal{S}^{new}$ with $ji \in \mathcal{O}^{new}$. With (4.4.2), we get $ij \in \mathcal{O}^{old}$ and, therefore, $j \notin \mathcal{S}^{old}$. With $j \notin \mathcal{S}^{old}$ and $j \in \mathcal{S}^{new}$, we get the "$\Rightarrow$" direction of (4.4.3). The proof for the "$\Leftarrow$" direction of (4.4.3) is analogous.

<u>Part 2:</u>

Suppose $\mathcal{S}^{old} = A$. Then we get $\mathcal{O}^{old} = \emptyset$. Otherwise, if there was an $ij \in \mathcal{O}^{old}$, we would immediately get $j \notin \mathcal{S}^{old}$ and, therefore, $\mathcal{S}^{old} \neq A$. With $\mathcal{O}^{old} = \emptyset$ and (4.4.2), we get $\mathcal{O}^{new} = \emptyset$ and, therefore, $\mathcal{S}^{new} = A$. With $\mathcal{S}^{old} = A$ and $\mathcal{S}^{new} = A$, we get $\mathcal{S}^{old} = \mathcal{S}^{new}$.

<u>Part 3:</u>

Suppose $\mathcal{S}^{old} \neq A$. Then there is a $j \notin \mathcal{S}^{old}$. With (4.1.14), we get that there is an $i \in \mathcal{S}^{old}$ with $ij \in \mathcal{O}^{old}$. With (4.4.2), we get $ji \in \mathcal{O}^{new}$ and, therefore, $i \notin \mathcal{S}^{new}$. With $i \in \mathcal{S}^{old}$ and $i \notin \mathcal{S}^{new}$, we get $\mathcal{S}^{old} \neq \mathcal{S}^{new}$. With part 2 and part 3, we get (4.4.4). □

4.5. Monotonicity

Monotonicity says that, when some voters rank alternative $a \in A$ higher [see (4.5.1) and (4.5.2)] without changing the order in which they rank the other alternatives relatively to each other [see (4.5.3)], then this must not hurt alternative *a* [see (4.5.4) – (4.5.6)]. Monotonicity is also known as *mono-raise* and *non-negative responsiveness*.

Definition:

An election method satisfies *monotonicity* if the following holds:

Suppose $a \in A$. Suppose the ballots are modified in such a manner that the following three statements are satisfied:

(4.5.1) $\forall f \in A \setminus \{a\} \; \forall v \in V$: $a \succ_v^{\text{old}} f \Rightarrow a \succ_v^{\text{new}} f$.

(4.5.2) $\forall f \in A \setminus \{a\} \; \forall v \in V$: $a \succsim_v^{\text{old}} f \Rightarrow a \succsim_v^{\text{new}} f$.

(4.5.3) $\forall e,f \in A \setminus \{a\} \; \forall v \in V$: $e \succ_v^{\text{old}} f \Leftrightarrow e \succ_v^{\text{new}} f$.

Then:

(4.5.4) $\forall b \in A \setminus \{a\}$: $ab \in \mathcal{O}^{\text{old}} \Rightarrow ab \in \mathcal{O}^{\text{new}}$.

(4.5.5) $\forall b \in A \setminus \{a\}$: $ba \notin \mathcal{O}^{\text{old}} \Rightarrow ba \notin \mathcal{O}^{\text{new}}$.

(4.5.6) $a \in \mathcal{S}^{\text{old}} \Rightarrow a \in \mathcal{S}^{\text{new}} \subseteq \mathcal{S}^{\text{old}}$.

Claim:

If $\succ_D$ satisfies (2.1.1), then the Schulze method, as defined in section 2.2, satisfies monotonicity.

Proof:

Part 1:

With (4.5.1), we get

(4.5.7) $\forall f \in A \setminus \{a\}$: $N^{\text{old}}[a,f] \leq N^{\text{new}}[a,f]$.

With (4.5.2), we get

(4.5.8) $\forall f \in A \setminus \{a\}$: $N^{\text{old}}[f,a] \geq N^{\text{new}}[f,a]$.

With (4.5.3), we get

(4.5.9) $\forall e,f \in A \setminus \{a\}$: $N^{\text{old}}[e,f] = N^{\text{new}}[e,f]$.

With (2.1.1), (4.5.7), and (4.5.8), we get

(4.5.10) $\forall f \in A \setminus \{a\}$: $(N^{\text{old}}[a,f],N^{\text{old}}[f,a]) \precsim_D (N^{\text{new}}[a,f],N^{\text{new}}[f,a])$.

With (2.1.1), (4.5.7), and (4.5.8), we get

(4.5.11) $\forall\, f \in A \setminus \{a\}$: $(N^{\text{old}}[f,a],N^{\text{old}}[a,f]) \gtrsim_D (N^{\text{new}}[f,a],N^{\text{new}}[a,f])$.

With (4.5.9), we get

(4.5.12) $\forall\, e,f \in A \setminus \{a\}$: $(N^{\text{old}}[e,f],N^{\text{old}}[f,e]) \approx_D (N^{\text{new}}[e,f],N^{\text{new}}[f,e])$.

Suppose $c(1),...,c(n) \in A$ was the strongest path from alternative a to alternative $b \in A \setminus \{a\}$. Then with (4.5.10) and (4.5.12), we get: $c(1),...,c(n)$ is a path from alternative a to alternative b with at least the same strength. Therefore

(4.5.13) $\forall\, b \in A \setminus \{a\}$: $P_D^{\text{new}}[a,b] \gtrsim_D P_D^{\text{old}}[a,b]$.

Suppose $c(1),...,c(n) \in A$ is the strongest path from alternative $b \in A \setminus \{a\}$ to alternative a. Then with (4.5.11) and (4.5.12), we get: $c(1),...,c(n)$ was a path from alternative b to alternative a with at least the same strength. Therefore

(4.5.14) $\forall\, b \in A \setminus \{a\}$: $P_D^{\text{old}}[b,a] \gtrsim_D P_D^{\text{new}}[b,a]$.

With (4.5.13) and (4.5.14), we get (4.5.4) and (4.5.5).

<u>Part 2:</u>

It remains to prove (4.5.6). Suppose $a \in \mathcal{S}^{\text{old}}$. Then " $a \in \mathcal{S}^{\text{new}}$ " follows directly from (4.5.5). To prove " $\mathcal{S}^{\text{new}} \subseteq \mathcal{S}^{\text{old}}$ ", we have to prove: $h \notin \mathcal{S}^{\text{old}} \Rightarrow h \notin \mathcal{S}^{\text{new}}$.

As $a \in \mathcal{S}^{\text{old}}$, we get

(4.5.15) $\forall\, b \in A \setminus \{a\}$: $P_D^{\text{old}}[a,b] \gtrsim_D P_D^{\text{old}}[b,a]$.

Suppose $h \notin \mathcal{S}^{\text{old}}$. Then, according to (4.1.14), there must have been an alternative $g \in \mathcal{S}^{\text{old}}$ with

(4.5.16) $P_D^{\text{old}}[g,h] \succ_D P_D^{\text{old}}[h,g]$.

With (4.5.10) – (4.5.12) and (4.5.16), we get: $P_D^{\text{new}}[g,h] \succ_D P_D^{\text{new}}[h,g]$, unless at least one of the following two cases occurred.

Case 1: xa was a weakest link in the strongest path from alternative g to alternative h.

Case 2: ay was the weakest link in the strongest path from alternative h to alternative g.

With (4.5.15), we get: $P_D^{\text{old}}[a,h] \gtrsim_D P_D^{\text{old}}[h,a]$. For $P_D^{\text{old}}[a,h] \succ_D P_D^{\text{old}}[h,a]$, we would, with (4.5.4), immediately get $P_D^{\text{new}}[a,h] \succ_D P_D^{\text{new}}[h,a]$, so that alternative h is still not a potential winner. Therefore, without loss of generality, we can presume $g \in \mathcal{S}^{\text{old}} \setminus \{a\}$ and

(4.5.17) $P_D^{\text{old}}[a,h] \approx_D P_D^{\text{old}}[h,a]$.

With $a \in \mathcal{S}^{\text{old}}$ and $g \in \mathcal{S}^{\text{old}} \setminus \{a\}$, we get

(4.5.18) $P_D^{\text{old}}[a,g] \approx_D P_D^{\text{old}}[g,a]$.

With (2.2.5), we get

(4.5.19) $\min_D \{ P_D^{\text{old}}[g,h], P_D^{\text{old}}[h,a] \} \precsim_D P_D^{\text{old}}[g,a]$.

(4.5.20) $\min_D \{ P_D^{\text{old}}[h,a], P_D^{\text{old}}[a,g] \} \precsim_D P_D^{\text{old}}[h,g]$.

<u>Case 1:</u> Suppose *xa* was a weakest link in the strongest path from alternative *g* to alternative *h*. Then

(4.5.21a) $P_D^{\text{old}}[g,h] \approx_D P_D^{\text{old}}[g,a]$ and

(4.5.22a) $P_D^{\text{old}}[a,h] \succsim_D P_D^{\text{old}}[g,h]$.

Now (4.5.18), (4.5.21a), and (4.5.16) give

(4.5.23a) $P_D^{\text{old}}[a,g] \approx_D P_D^{\text{old}}[g,a] \approx_D P_D^{\text{old}}[g,h] \succ_D P_D^{\text{old}}[h,g]$,

while (4.5.17), (4.5.22a), and (4.5.16) give

(4.5.24a) $P_D^{\text{old}}[h,a] \approx_D P_D^{\text{old}}[a,h] \succsim_D P_D^{\text{old}}[g,h] \succ_D P_D^{\text{old}}[h,g]$.

But (4.5.23a) and (4.5.24a) together contradict (4.5.20).

<u>Case 2:</u> Suppose *ay* was the weakest link in the strongest path from alternative *h* to alternative *g*. Then

(4.5.21b) $P_D^{\text{old}}[h,g] \approx_D P_D^{\text{old}}[a,g]$ and

(4.5.22b) $P_D^{\text{old}}[h,a] \succ_D P_D^{\text{old}}[h,g]$.

Now (4.5.22b), (4.5.21b), and (4.5.18) give

(4.5.23b) $P_D^{\text{old}}[h,a] \succ_D P_D^{\text{old}}[h,g] \approx_D P_D^{\text{old}}[a,g] \approx_D P_D^{\text{old}}[g,a]$,

while (4.5.16), (4.5.21b), and (4.5.18) give

(4.5.24b) $P_D^{\text{old}}[g,h] \succ_D P_D^{\text{old}}[h,g] \approx_D P_D^{\text{old}}[a,g] \approx_D P_D^{\text{old}}[g,a]$.

But (4.5.23b) and (4.5.24b) together contradict (4.5.19).

We have proven that neither case 1 nor case 2 is possible. Therefore

(4.5.25) $P_D^{\text{new}}[g,h] \succ_D P_D^{\text{new}}[h,g]$.

With (4.5.25), we get: $h \notin \mathcal{S}^{\text{new}}$. □

4.6. Independence of Clones

Independence of clones as a criterion for single-winner election methods has been proposed by Tideman (1987). This criterion says that running a large number of similar alternatives, so-called *clones*, must not have any impact on the result of the elections.

The precise definition for a *set of clones* stipulates that every voters ranks all the alternatives of this set in a consecutive manner; see (4.6.1) and (4.6.2). Replacing an alternative $d \in A^{\text{old}}$ by a set of clones K should not change the winner; see (4.6.7) and (4.6.8).

This criterion is very desirable especially for referendums because, while it might be difficult to find several candidates who are simultaneously sufficiently popular to campaign with them and sufficiently similar to misuse them for this strategy, it is usually very simple to formulate a large number of almost identical proposals. For example: In 1969, when the Canadian city that is now known as *Thunder Bay* was amalgamating, there was some controversy over what the name should be. In opinion polls, a majority of the voters preferred the name *The Lakehead* to the name *Thunder Bay*. But when the polls opened, there were three names on the referendum ballot: *Thunder Bay*, *Lakehead*, and *The Lakehead*. As the ballots were counted using *plurality voting*, it was not a surprise when *Thunder Bay* won. The votes were as follows: *Thunder Bay* 15870, *Lakehead* 15302, *The Lakehead* 8377 (Cretney, 2000).

Definition:

An election method is *independent of clones* if the following holds:

Suppose $d \in A^{\text{old}}$. Suppose $A^{\text{new}} := (A^{\text{old}} \cup K) \setminus \{d\}$.

Suppose alternative d is replaced by the set of alternatives K in such a manner that the following three statements are satisfied:

(4.6.1) $\forall\, e \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K\ \forall\, v \in V$: $e \succ_v^{\text{old}} d \Leftrightarrow e \succ_v^{\text{new}} g$.

(4.6.2) $\forall\, f \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K\ \forall\, v \in V$: $d \succ_v^{\text{old}} f \Leftrightarrow g \succ_v^{\text{new}} f$.

(4.6.3) $\forall\, e,f \in A^{\text{old}} \setminus \{d\}\ \forall\, v \in V$: $e \succ_v^{\text{old}} f \Leftrightarrow e \succ_v^{\text{new}} f$.

Then the following statements are satisfied:

(4.6.4) $\forall\, a \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K$: $ad \in \mathcal{O}^{\text{old}} \Leftrightarrow ag \in \mathcal{O}^{\text{new}}$.

(4.6.5) $\forall\, b \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K$: $db \in \mathcal{O}^{\text{old}} \Leftrightarrow gb \in \mathcal{O}^{\text{new}}$.

(4.6.6) $\forall\, a,b \in A^{\text{old}} \setminus \{d\}$: $ab \in \mathcal{O}^{\text{old}} \Leftrightarrow ab \in \mathcal{O}^{\text{new}}$.

(4.6.7) $d \in \mathcal{S}^{\text{old}} \Leftrightarrow \mathcal{S}^{\text{new}} \cap K \neq \emptyset$.

(4.6.8) $\forall\, a \in A^{\text{old}} \setminus \{d\}$: $a \in \mathcal{S}^{\text{old}} \Leftrightarrow a \in \mathcal{S}^{\text{new}}$.

Claim:

The Schulze method, as defined in section 2.2, is independent of clones.

Proof:

With (4.6.1), we get

(4.6.9) $\forall\, e \in A^{old} \setminus \{d\}\ \forall\, g \in K$: $N^{old}[e,d] = N^{new}[e,g]$.

With (4.6.2), we get

(4.6.10) $\forall\, f \in A^{old} \setminus \{d\}\ \forall\, g \in K$: $N^{old}[d,f] = N^{new}[g,f]$.

With (4.6.3), we get

(4.6.11) $\forall\, e,f \in A^{old} \setminus \{d\}$: $N^{old}[e,f] = N^{new}[e,f]$.

With (4.6.9) and (4.6.10), we get

(4.6.12) $\forall\, e \in A^{old} \setminus \{d\}\ \forall\, g \in K$: $(N^{old}[e,d],N^{old}[d,e]) \approx_D (N^{new}[e,g],N^{new}[g,e])$.

With (4.6.9) and (4.6.10), we get

(4.6.13) $\forall\, f \in A^{old} \setminus \{d\}\ \forall\, g \in K$: $(N^{old}[d,f],N^{old}[f,d]) \approx_D (N^{new}[g,f],N^{new}[f,g])$.

With (4.6.11), we get

(4.6.14) $\forall\, e,f \in A^{old} \setminus \{d\}$: $(N^{old}[e,f],N^{old}[f,e]) \approx_D (N^{new}[e,f],N^{new}[f,e])$.

Suppose $c(1),...,c(n) \in A^{old}$ was the strongest path from alternative $a \in A^{old} \setminus \{d\}$ to alternative d. Then with (4.6.12) and (4.6.14), we get: $c(1),...,c(n-1),g$ is a path from alternative a to alternative $g \in K$ with the same strength. Therefore

(4.6.15) $\forall\, a \in A^{old} \setminus \{d\}\ \forall\, g \in K$: $P_D^{new}[a,g] \succsim_D P_D^{old}[a,d]$.

Suppose $c(1),...,c(n) \in A^{new}$ is the strongest path from alternative $a \in A^{new} \setminus K$ to alternative $g \in K$. Then with (4.6.12) and (4.6.14), we get: $c(1),...,c(m-1),d$, where $c(m)$ is the first occurrence of an alternative of the set K, was a path from alternative a to alternative d with at least the same strength. Therefore

(4.6.16) $\forall\, a \in A^{new} \setminus K\ \forall\, g \in K$: $P_D^{old}[a,d] \succsim_D P_D^{new}[a,g]$.

Suppose $c(1),...,c(n) \in A^{old}$ was the strongest path from alternative d to alternative $b \in A^{old} \setminus \{d\}$. Then with (4.6.13) and (4.6.14), we get: $g,c(2),...,c(n)$ is a path from alternative $g \in K$ to alternative b with the same strength. Therefore

(4.6.17) $\forall\, b \in A^{old} \setminus \{d\}\ \forall\, g \in K$: $P_D^{new}[g,b] \succsim_D P_D^{old}[d,b]$.

Suppose $c(1),...,c(n) \in A^{new}$ is the strongest path from alternative $g \in K$ to alternative $b \in A^{new} \setminus K$. Then with (4.6.13) and (4.6.14), we get: $d,c(m+1),...,c(n)$, where $c(m)$ is the last occurrence of an alternative of the set K, was a path from alternative d to alternative b with at least the same strength. Therefore

(4.6.18) $\forall\, b \in A^{new} \setminus K\ \forall\, g \in K$: $P_D^{old}[d,b] \succsim_D P_D^{new}[g,b]$.

(α) Suppose the strongest path $c(1),...,c(n) \in A^{\text{old}}$ from alternative $a \in A^{\text{old}} \setminus \{d\}$ to alternative $b \in A^{\text{old}} \setminus \{a,d\}$ did not contain alternative d. Then with (4.6.14), we get: $c(1),...,c(n)$ is still a path from alternative a to alternative b with the same strength. Therefore: $P_D^{\text{new}}[a,b] \gtrsim_D P_D^{\text{old}}[a,b]$.

(β) Suppose the strongest path $c(1),...,c(n) \in A^{\text{old}}$ from alternative $a \in A^{\text{old}} \setminus \{d\}$ to alternative $b \in A^{\text{old}} \setminus \{a,d\}$ contained alternative d. Then with (4.6.12), (4.6.13), and (4.6.14), we get: $c(1),...,c(n)$, with alternative d replaced by an arbitrarily chosen alternative $g \in K$, is still a path from alternative a to alternative b with the same strength. Therefore: $P_D^{\text{new}}[a,b] \gtrsim_D P_D^{\text{old}}[a,b]$.

With (α) and (β), we get

(4.6.19) $\quad \forall\, a,b \in A^{\text{old}} \setminus \{d\}$: $P_D^{\text{new}}[a,b] \gtrsim_D P_D^{\text{old}}[a,b]$.

(γ) Suppose the strongest path $c(1),...,c(n) \in A^{\text{new}}$ from alternative $a \in A^{\text{new}} \setminus K$ to alternative $b \in A^{\text{new}} \setminus (\, K \cup \{a\}\,)$ does not contain alternatives of the set K. Then with (4.6.14), we get: $c(1),...,c(n)$ was a path from alternative a to alternative b with the same strength. Therefore: $P_D^{\text{old}}[a,b] \gtrsim_D P_D^{\text{new}}[a,b]$.

(δ) Suppose the strongest path $c(1),...,c(n) \in A^{\text{new}}$ from alternative $a \in A^{\text{new}} \setminus K$ to alternative $b \in A^{\text{new}} \setminus (\, K \cup \{a\}\,)$ contains some alternatives of the set K. Then with (4.6.12), (4.6.13), and (4.6.14), we get: $c(1),...,c(s-1),d,c(t+1),...,c(n)$, where $c(s)$ is the first occurrence of an alternative of the set K and $c(t)$ is the last occurrence of an alternative of the set K, was a path from alternative a to alternative b with at least the same strength. Therefore: $P_D^{\text{old}}[a,b] \gtrsim_D P_D^{\text{new}}[a,b]$.

With (γ) and (δ), we get

(4.6.20) $\quad \forall\, a,b \in A^{\text{new}} \setminus K$: $P_D^{\text{old}}[a,b] \gtrsim_D P_D^{\text{new}}[a,b]$.

Combining (4.6.15) and (4.6.16) gives

(4.6.21) $\quad \forall\, a \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K$: $P_D^{\text{old}}[a,d] \approx_D P_D^{\text{new}}[a,g]$.

Combining (4.6.17) and (4.6.18) gives

(4.6.22) $\quad \forall\, b \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K$: $P_D^{\text{old}}[d,b] \approx_D P_D^{\text{new}}[g,b]$.

Combining (4.6.19) and (4.6.20) gives

(4.6.23) $\quad \forall\, a,b \in A^{\text{old}} \setminus \{d\}$: $P_D^{\text{old}}[a,b] \approx_D P_D^{\text{new}}[a,b]$.

With (4.6.21) – (4.6.23), we get (4.6.4) – (4.6.6).

Part 1:

Suppose $d \in \mathcal{S}^{\text{old}}$. Then

(4.6.24) $\quad \forall\, a \in A^{\text{old}} \setminus \{d\}$: $ad \notin \mathcal{O}^{\text{old}}$.

With (4.6.4) and (4.6.24), we get

(4.6.25) $\forall\ a \in A^{new} \setminus K\ \forall\ g \in K$: $ag \notin \mathcal{O}^{new}$.

Since the binary relation $\mathcal{O}^{new}$, as defined in (2.2.1), is asymmetric and transitive, there must be an alternative $k \in K$ with

(4.6.26) $\forall\ l \in K \setminus \{k\}$: $lk \notin \mathcal{O}^{new}$.

With (4.6.25) and (4.6.26), we get $k \in \mathcal{S}^{new} \cap K$ and, therefore, $\mathcal{S}^{new} \cap K \neq \emptyset$.

Part 2:

Suppose $d \notin \mathcal{S}^{old}$. Then

(4.6.27) $\exists\ a \in A^{old} \setminus \{d\}$: $ad \in \mathcal{O}^{old}$.

With (4.6.4) and (4.6.27), we get

(4.6.28) $\exists\ a \in A^{new} \setminus K\ \forall\ g \in K$: $ag \in \mathcal{O}^{new}$.

With (4.6.28), we get: $\mathcal{S}^{new} \cap K = \emptyset$.

With part 1 and part 2, we get (4.6.7).

Part 3:

Suppose $a \in A^{old} \setminus \{d\}$ and $a \in \mathcal{S}^{old}$. Then

(4.6.29) $da \notin \mathcal{O}^{old}$.

(4.6.30) $\forall\ b \in A^{old} \setminus \{a,d\}$: $ba \notin \mathcal{O}^{old}$.

With (4.6.5) and (4.6.29), we get

(4.6.31) $\forall\ g \in K$: $ga \notin \mathcal{O}^{new}$.

With (4.6.6) and (4.6.30), we get

(4.6.32) $\forall\ b \in A^{new} \setminus (\ K \cup \{a\}\ )$: $ba \notin \mathcal{O}^{new}$.

With (4.6.31) and (4.6.32), we get: $a \in \mathcal{S}^{new}$.

Part 4:

Suppose $a \in A^{old} \setminus \{d\}$ and $a \notin \mathcal{S}^{old}$. Then at least one of the following two statements must have been valid:

(4.6.33a) $da \in \mathcal{O}^{old}$.

(4.6.33b) $\exists\ b \in A^{old} \setminus \{a,d\}$: $ba \in \mathcal{O}^{old}$.

With (4.6.5), (4.6.6), and (4.6.33), we get that at least one of the following two statements must be valid:

(4.6.34a) $\forall\ g \in K$: $ga \in \mathcal{O}^{new}$.

(4.6.34b) $\exists\ b \in A^{new} \setminus (\ K \cup \{a\}\ )$: $ba \in \mathcal{O}^{new}$.

With (4.6.34), we get: $a \notin \mathcal{S}^{new}$.

With part 3 and part 4, we get (4.6.8). □

4.7. Smith

The *Smith criterion* and *Smith-IIA* (where IIA means "independence of irrelevant alternatives") say that *weak* alternatives should have no impact on the result of the elections.

Suppose:

(4.7.1) $\emptyset \neq B_1 \subsetneq A$, $\emptyset \neq B_2 \subsetneq A$, $B_1 \cup B_2 = A$, $B_1 \cap B_2 = \emptyset$.

(4.7.2) $\forall\ a \in B_1\ \forall\ b \in B_2$: $N[a,b] > N[b,a]$.

Then a *weak* alternative in the Smith paradigm is an alternative $b \in B_2$. Adding or removing a weak alternative $b \in B_2$ should have no impact on the set $\mathcal{S}$ of winners.

<u>Definition:</u>

An election method satisfies the *Smith criterion* if the following holds:

Suppose (4.7.1) and (4.7.2). Then:

(4.7.3) $\forall\ a \in B_1\ \forall\ b \in B_2$: $ab \in \mathcal{O}$.

(4.7.4) $\emptyset \neq \mathcal{S} \subseteq B_1$.

<u>Remark:</u>

If B_1 consists of only one alternative $a \in A$, then this is the so-called *Condorcet criterion* (Condorcet, 1785). If B_2 consists of only one alternative $b \in A$, then this is the so-called *Condorcet loser criterion*.

<u>Claim:</u>

If $\succ_D$ satisfies (2.1.5), then the Schulze method, as defined in section 2.2, satisfies the Smith criterion.

<u>Proof:</u>

The proof is trivial. Presumption (2.1.5) guarantees that any pairwise victory is stronger than any pairwise defeat. If $a \in B_1$ and $b \in B_2$, then already the link *ab* is a path from alternative *a* to alternative *b* that consists only of a pairwise victory. On the other side, (4.7.2) says that there cannot be a path from alternative *b* to alternative *a* that contains no pairwise defeat. So already the link *ab* is stronger than any path from alternative *b* to alternative *a*. □

Definition:

An election method satisfies *Smith-IIA* if the following holds:

Suppose (4.7.1) and (4.7.2). Then:

(4.7.5) If $d \in B_2$ is removed, then

(a) $\forall\ e,f \in B_1$: $ef \in \mathcal{O}^{old} \Leftrightarrow ef \in \mathcal{O}^{new}$.

(b) $\mathcal{S}^{old} = \mathcal{S}^{new}$.

(4.7.6) If $d \in B_1$ is removed, then
$\forall\ e,f \in B_2$: $ef \in \mathcal{O}^{old} \Leftrightarrow ef \in \mathcal{O}^{new}$.

Claim:

If $\succ_D$ satisfies (2.1.5), then the Schulze method, as defined in section 2.2, satisfies Smith-IIA.

Proof:

We will prove (4.7.5)(a). The proof for (4.7.6) is analogous.

(4.7.5)(b) follows directly from (4.7.4) and (4.7.5)(a).

Part 1: Suppose $e,f \in B_1$. Suppose $ef \in \mathcal{O}^{old}$. Then

(4.7.7) $P^{old}_D[e,f] \succ_D P^{old}_D[f,e]$.

With (2.2.3), we get

(4.7.8) $P^{old}_D[e,f] \succsim_D (N[e,f],N[f,e])$.

With (4.7.7) and (2.2.3), we get

(4.7.9) $P^{old}_D[e,f] \succ_D P^{old}_D[f,e] \succsim_D (N[f,e],N[e,f])$.

With (4.7.8) and (4.7.9), we get

(4.7.10) $P^{old}_D[e,f] \succsim_D \max_D \{ (N[e,f],N[f,e]), (N[f,e],N[e,f]) \}$.

With (4.7.2), we get: Any path from alternative $e \in B_1$ to alternative $f \in B_1$ that contained alternative $d \in B_2$ necessarily contained a pairwise defeat.

As it is not possible that the link ef is a pairwise defeat and that simultaneously the link fe is a pairwise defeat, $\max_D \{ (N[e,f],N[f,e]), (N[f,e],N[e,f]) \}$ is stronger than any pairwise defeat [because of (2.1.5)]. Therefore, with (4.7.2) and (4.7.10), we get: The strongest path from alternative $e \in B_1$ to alternative $f \in B_1$ did not contain alternative $d \in B_2$. Therefore

(4.7.11) $P^{new}_D[e,f] \approx_D P^{old}_D[e,f]$.

As the elimination of alternative $d \in B_2$ only removes paths, we get

(4.7.12) $\quad P_D^{\text{new}}[f,e] \precsim_D P_D^{\text{old}}[f,e]$.

With (4.7.11), (4.7.7), and (4.7.12), we get

(4.7.13) $\quad P_D^{\text{new}}[e,f] \approx_D P_D^{\text{old}}[e,f] \succ_D P_D^{\text{old}}[f,e] \succsim_D P_D^{\text{new}}[f,e]$.

With (4.7.13), we get: $ef \in \mathcal{O}^{\text{new}}$.

<u>Part 2:</u> The proof “ $ef \notin \mathcal{O}^{\text{old}} \Rightarrow ef \notin \mathcal{O}^{\text{new}}$ ” is analogous. □

The *majority criterion for solid coalitions* says that, when a majority of the voters strictly prefers every alternative of a given set of alternatives to every alternative outside this set of alternatives, then the winner must be chosen from this set. In short, an election method satisfies the *majority criterion for solid coalitions* if the following holds:

Suppose (4.7.1).
Suppose $\| \{ v \in V \mid \forall\, a \in B_1\ \forall\, b \in B_2: a \succ_v b \} \| > N/2$.
Then $\mathcal{S} \subseteq B_1$.

If B_1 consists of only one alternative $a \in A$, then this is the so-called *majority criterion*. If B_2 consists of only one alternative $b \in A$, then this is the so-called *majority loser criterion*.

Participation says that adding a list W of ballots, on which every alternative of a given set of alternatives is strictly preferred to every alternative outside this set, must not hurt the alternatives of this set. In short, an election method satisfies *participation* if the following holds:

Suppose (4.7.1).
Suppose $\forall\, a \in B_1\ \forall\, b \in B_2\ \forall\, w \in W: a \succ_w b$.
Suppose $V^{\text{new}} := V^{\text{old}} + W$.

Then
(4.7.14) $\quad \forall\, e \in B_1\ \forall\, f \in B_2: ef \in \mathcal{O}^{\text{old}} \Rightarrow ef \in \mathcal{O}^{\text{new}}$.
(4.7.15) $\quad \forall\, e \in B_1\ \forall\, f \in B_2: fe \notin \mathcal{O}^{\text{old}} \Rightarrow fe \notin \mathcal{O}^{\text{new}}$.
(4.7.16) $\quad \mathcal{S}^{\text{old}} \cap B_1 \neq \emptyset \Rightarrow \mathcal{S}^{\text{new}} \cap B_1 \neq \emptyset$.
(4.7.17) $\quad \mathcal{S}^{\text{old}} \cap B_2 = \emptyset \Rightarrow \mathcal{S}^{\text{new}} \cap B_2 = \emptyset$.

The Smith criterion implies the majority criterion for solid coalitions, the Condorcet criterion, and the Condorcet loser criterion. The majority criterion for solid coalitions implies the majority criterion and the majority loser criterion. The Condorcet criterion implies the majority criterion. The Condorcet loser criterion implies the majority loser criterion. Unfortunately, the Condorcet criterion is incompatible with the participation criterion (Moulin, 1988). Example 5 shows a drastic violation of the participation criterion.

4.8. MinMax Set

For all $\emptyset \neq B \subsetneq A$, we define

(4.8.1) $\Gamma_D(B) := \max_D \{ (N[x,y],N[y,x]) \mid x \notin B, y \in B \}$.

Furthermore, we define

(4.8.2) $\beta_D := \min_D \{ \Gamma_D(B) \mid \emptyset \neq B \subsetneq A \}$.

(4.8.3) $\mathfrak{B}_D := \cup \{ \emptyset \neq B \subsetneq A \mid \Gamma_D(B) \approx_D \beta_D \}$.

$\mathfrak{B}_D$ is the *MinMax set*. $\mathfrak{B}_D$ has the following properties:

1. $\mathfrak{B}_D \neq \emptyset$.
2. If $\mathfrak{B}_D$ consists of only one alternative $a \in A$, then alternative a is the unique Simpson-Kramer winner (i.e. that alternative $a \in A$ with minimum $\max_D \{ (N[b,a],N[a,b]) \mid b \in A \setminus \{a\} \}$).
3. If $d \in \mathfrak{B}_D$ is replaced by a set of alternatives K as described in (4.6.1) – (4.6.3), then $\mathfrak{B}_D^{new} = (\mathfrak{B}_D \cup K) \setminus \{d\}$.
4. If $d \notin \mathfrak{B}_D$ is replaced by a set of alternatives K as described in (4.6.1) – (4.6.3), then $\mathfrak{B}_D^{new} = \mathfrak{B}_D$.

So, in some sense, the MinMax set $\mathfrak{B}_D$ is a clone-proof generalization of the Simpson-Kramer winner.

When we want primarily that the used election method is independent of clones and secondarily that the strongest link *ef*, that is overruled when determining the winner, is minimized, then we have to demand that the winner is always chosen from the MinMax set $\mathfrak{B}_D$.

Claim:

The Schulze method, as defined in section 2.2, has the following properties:

(4.8.4) $\forall a \in \mathfrak{B}_D \; \forall b \notin \mathfrak{B}_D$: $ab \in \mathcal{O}$.

(4.8.5) $\mathcal{S} \subseteq \mathfrak{B}_D$.

Proof:

Suppose $a \in \mathfrak{B}_D$. Then we get

(4.8.6) $\exists \emptyset \neq B \subsetneq A$: $\Gamma_D(B) \approx_D \beta_D$ and $a \in B$.

Suppose $b \notin \mathfrak{B}_D$. Then we get

(4.8.7) $\gamma_D := \min_D \{ \Gamma_D(B) \mid \emptyset \neq B \subsetneq A \text{ and } b \in B \} \succ_D \beta_D$.

We will prove the following claims:

Claim #1: $P_D[b,a] \precsim_D \beta_D$.
Claim #2: $P_D[a,b] \succsim_D \gamma_D$.

With claim #1, claim #2, and (4.8.7), we get

(4.8.8) $\quad P_D[a,b] \succsim_D \gamma_D \succ_D \beta_D \succsim_D P_D[b,a]$.

With (4.8.8), we get (4.8.4). With (4.8.4), we get (4.8.5).

Proof of claim #1:

With (4.8.6) and (4.8.7), we get

(4.8.9) $\quad \exists\, \varnothing \neq B \subsetneq A$: $\Gamma_D(B) \approx_D \beta_D$ and $a \in B$ and $b \notin B$.

Suppose $c(1),\ldots,c(n) \in A$ is the strongest path from alternative b to alternative a. Suppose $c(i)$ is the last alternative with $c(i) \notin B$. Then we get $(N[c(i),c(i+1)],N[c(i+1),c(i)]) \precsim_D \beta_D$. Therefore, we get

(4.8.10) $\quad P_D[b,a] \precsim_D \beta_D$.

Proof of claim #2:

We can construct a path from alternative a to alternative b with a strength of at least γ_D as follows:

(1) We start with $E_1 := \{a\}$ and $i := 1$. Trivially, we get $b \notin E_1$ and $P_D[a,h] \succsim_D \gamma_D$ for all $h \in E_1 \setminus \{a\}$.

(2) At each stage, we consider the set $B_i := A \setminus E_i$.

With $b \in B_i$ and with (4.8.7), we get

(4.8.11) $\quad \Gamma_D(B_i) \approx_D \max_D \{ (N[y,x],N[x,y]) \mid y \notin B_i, x \in B_i \} \succsim_D \gamma_D$.

We choose $f \in E_i$ and $g \in B_i$ with

(4.8.12) $\quad (N[f,g],N[g,f]) \approx_D \max_D \{ (N[y,x],N[x,y]) \mid y \notin B_i, x \in B_i \} \succsim_D \gamma_D$.

We define $E_{i+1} := E_i \cup \{g\}$.

With $f \in E_i$, with $P_D[a,h] \succsim_D \gamma_D$ for all $h \in E_i \setminus \{a\}$, with $(N[f,g], N[g,f]) \succsim_D \gamma_D$, and with $E_{i+1} := E_i \cup \{g\}$, we get

(4.8.13) $\quad P_D[a,h] \succsim_D \gamma_D$ for all $h \in E_{i+1} \setminus \{a\}$.

(3) We repeat stage 2 with $i \to i+1$, until $g \equiv b$.

Therefore, we get

(4.8.14) $\quad P_D[a,b] \succsim_D \gamma_D$. $\square$

Example 6 shows that IPDA and the desideratum, that the winner is always chosen from the MinMax set $\mathfrak{B}_D$, are incompatible. In example 6(old), we get $\mathfrak{B}_D^{old} = \{a, c, d\}$. In example 6(new), we get $\mathfrak{B}_D^{new} = \{b\}$. Therefore, $\mathfrak{B}_D^{old} \cap \mathfrak{B}_D^{new} = \varnothing$. Thus, the desideratum, that the winner is always chosen from the MinMax set $\mathfrak{B}_D$, implies that the winner is changed.

Actually, the Schulze method can be described completely with the desideratum to find a binary relation $\mathcal{O}$ on A that, primarily, is independent of clones (as defined in section 4.6) and that, secondarily, tries to rank the alternatives according to their worst defeats.

For all $a,b \in A$, we define

(4.8.15) $\gamma_D[a,b] := \min_D \{ \Gamma_D(B) \mid \varnothing \neq B \subsetneq A \text{ and } a \notin B \text{ and } b \in B \}$.

(4.8.16) $ab \in \mathcal{O} :\Leftrightarrow \gamma_D[a,b] \succ_D \gamma_D[b,a]$.

To prove that (4.8.16) is identical to (2.2.1), we have to prove $\gamma_D[a,b] = P_D[a,b]$. This proof is identical to the proof for (4.8.4).

Example 1

In example 1 (section 3.1), we have:

$\Gamma_D(B) := \max_D \{ (N[x,y],N[y,x]) \mid x \notin B, y \in B \}$.

$\Gamma_D(\{a\}) = (13,8)$.
$\Gamma_D(\{b\}) = (19,2)$.
$\Gamma_D(\{c\}) = (14,7)$.
$\Gamma_D(\{d\}) = (12,9)$.
$\Gamma_D(\{a,b\}) = (19,2)$.
$\Gamma_D(\{a,c\}) = (13,8)$.
$\Gamma_D(\{a,d\}) = (13,8)$.
$\Gamma_D(\{b,c\}) = (19,2)$.
$\Gamma_D(\{b,d\}) = (15,6)$.
$\Gamma_D(\{c,d\}) = (14,7)$.
$\Gamma_D(\{a,b,c\}) = (19,2)$.
$\Gamma_D(\{a,b,d\}) = (15,6)$.
$\Gamma_D(\{a,c,d\}) = (13,8)$.
$\Gamma_D(\{b,c,d\}) = (14,7)$.

$\beta_D := \min_D \{ \Gamma_D(B) \mid \varnothing \neq B \subsetneq A \}$.

$\beta_D = (12,9)$.

$\mathfrak{B}_D := \cup \{ \varnothing \neq B \subsetneq A \mid \Gamma_D(B) \approx_D \beta_D \}$.

$\mathfrak{B}_D = \{d\}$.

So with (4.8.5), we get $\mathcal{S} = \{d\}$.

$\gamma_D[x,y] := \min_D \{ \Gamma_D(B) \mid \varnothing \neq B \subsetneq A \text{ and } x \notin B \text{ and } y \in B \}$.

$\gamma_D[a,b] = \Gamma_D(\{b,c,d\}) = (14,7)$.
$\gamma_D[a,c] = \Gamma_D(\{c\}) = \Gamma_D(\{c,d\}) = \Gamma_D(\{b,c,d\}) = (14,7)$.
$\gamma_D[a,d] = \Gamma_D(\{d\}) = (12,9)$.
$\gamma_D[b,a] = \Gamma_D(\{a\}) = \Gamma_D(\{a,c\}) = \Gamma_D(\{a,d\}) = \Gamma_D(\{a,c,d\}) = (13,8)$.
$\gamma_D[b,c] = \Gamma_D(\{a,c\}) = \Gamma_D(\{a,c,d\}) = (13,8)$.
$\gamma_D[b,d] = \Gamma_D(\{d\}) = (12,9)$.
$\gamma_D[c,a] = \Gamma_D(\{a\}) = \Gamma_D(\{a,d\}) = (13,8)$.
$\gamma_D[c,b] = \Gamma_D(\{b,d\}) = \Gamma_D(\{a,b,d\}) = (15,6)$.
$\gamma_D[c,d] = \Gamma_D(\{d\}) = (12,9)$.
$\gamma_D[d,a] = \Gamma_D(\{a\}) = \Gamma_D(\{a,c\}) = (13,8)$.
$\gamma_D[d,b] = \Gamma_D(\{b\}) = \Gamma_D(\{a,b\}) = \Gamma_D(\{b,c\}) = \Gamma_D(\{a,b,c\}) = (19,2)$.
$\gamma_D[d,c] = \Gamma_D(\{a,c\}) = (13,8)$.

4.9. Prudence

Prudence as a criterion for single-winner election methods has been proposed by Köhler (1978) and generalized by Arrow and Raynaud (1986). This criterion says that the strength λ_D of the strongest link ef, that is not respected by the binary relation $\mathcal{O}$, should be as weak as possible. So $\lambda_D := \max_D \{ (N[e,f],N[f,e]) \mid ef \notin \mathcal{O} \}$ should be minimized.

> A *directed cycle* is a sequence of alternatives $c(1),...,c(n) \in A$ with the following properties:
>
> 1. $c(1) \equiv c(n)$.
> 2. $n \in \mathbb{N}$ with $3 \leq n < \infty$.
> 3. For all $i = 1,...,(n-1)$: $c(i+1) \in A \setminus \{c(i)\}$.

It is obvious that, when there is a directed cycle $c(1),...,c(n)$, then the strongest link, that is not respected by the binary relation $\mathcal{O}$, is at least as strong as the weakest link $c(i),c(i+1)$ of this directed cycle. Therefore, we get:

(4.9.1) $\qquad \lambda_D \succsim_D \min_D \{ (N[c(i),c(i+1)],N[c(i+1),c(i)]) \mid i = 1,...,(n-1) \}$.

As we have to make this consideration for all directed cycles, the maximum, that we can ask for, is the following criterion.

<u>Definition:</u>

Suppose $\lambda_D \in \mathbb{N}_0 \times \mathbb{N}_0$ is the strength of the strongest directed cycle.

(4.9.2) $\qquad \lambda_D := \max_D \{ \min_D \{ (N[c(i),c(i+1)],N[c(i+1),c(i)]) \mid i = 1,...,(n-1) \} \mid c(1),...,c(n) \text{ is a directed cycle } \}$.

Then an election method is *prudent* if the following holds:

(4.9.3) $\qquad \forall\, a,b \in A: (N[a,b],N[b,a]) \succ_D \lambda_D \Rightarrow ab \in \mathcal{O}$.

(4.9.4) $\qquad \forall\, a,b \in A: (N[a,b],N[b,a]) \succ_D \lambda_D \Rightarrow b \notin \mathcal{S}$.

Claim:

The Schulze method, as defined in section 2.2, is prudent.

Proof:

The proof is trivial. With (2.2.4), we get: $ab \in \mathcal{O}$, unless the link ab is in a directed cycle that consists of links of which each is at least as strong as the link ab. □

Example 1

In example 1 (section 3.1), the strongest directed cycle (measured by the strength of its weakest link) is a,(14,7),c,(15,6),b,(13,8),a with a strength of $\lambda_D \approx_D (13,8)$. So prudence says that the collective ranking $\mathcal{O}$ must respect all links that are stronger than (13,8).

$(N[d,b],N[b,d]) = (19,2) \succ_D (13,8) \approx_D \lambda_D \Rightarrow db \in \mathcal{O}$.

$(N[c,b],N[b,c]) = (15,6) \succ_D (13,8) \approx_D \lambda_D \Rightarrow cb \in \mathcal{O}$.

$(N[a,c],N[c,a]) = (14,7) \succ_D (13,8) \approx_D \lambda_D \Rightarrow ac \in \mathcal{O}$.

With $db \in \mathcal{O}$, $cb \in \mathcal{O}$, and $ac \in \mathcal{O}$, we get $b \notin \mathcal{S}$ and $c \notin \mathcal{S}$.

4.10. Schwartz

The Schwartz criterion as a criterion for single-winner election methods has been proposed by Schwartz (1986). The Schwartz criterion implies the Smith criterion.

A *chain* from alternative $x \in A$ to alternative $y \in A$ is a sequence of alternatives $c(1),...,c(n) \in A$ with the following properties:

1. $x \equiv c(1)$.
2. $y \equiv c(n)$.
3. $2 \leq n < \infty$.
4. For all $i = 1,...,(n-1)$: $c(i+1) \in A \setminus \{c(i)\}$.
5. For all $i = 1,...,(n-1)$: $N[c(i),c(i+1)] > N[c(i+1),c(i)]$.

Definition:

An election method satisfies the *Schwartz criterion* if the following holds:

Suppose there is a chain from alternative $a \in A$ to alternative $b \in A$ and no chain from alternative b to alternative a. Then:

(4.10.1) $ab \in \mathcal{O}$.

(4.10.2) $b \notin \mathcal{S}$.

Claim:

If $\succ_D$ satisfies (2.1.5), then the Schulze method, as defined in section 2.2, satisfies the Schwartz criterion.

Proof:

The proof is trivial. □

4.11. Weak Condorcet Winners and Weak Condorcet Losers

A *Condorcet winner* is an alternative $a \in A$ that wins every head-to-head contest with some other alternative $b \in A \setminus \{a\}$. In other words:

(4.11.1) Alternative $a \in A$ is a *Condorcet winner* $:\Leftrightarrow$
$N[a,b] > N[b,a]$ for all $b \in A \setminus \{a\}$.

A *Condorcet loser* is an alternative $a \in A$ that loses every head-to-head contest with some other alternative $b \in A \setminus \{a\}$. In other words:

(4.11.2) Alternative $a \in A$ is a *Condorcet loser* $:\Leftrightarrow$
$N[a,b] < N[b,a]$ for all $b \in A \setminus \{a\}$.

A *weak Condorcet winner* is an alternative $a \in A$ that doesn't lose any head-to-head contest with some other alternative $b \in A \setminus \{a\}$. In other words:

(4.11.3) Alternative $a \in A$ is a *weak Condorcet winner* $:\Leftrightarrow$
$N[a,b] \geq N[b,a]$ for all $b \in A \setminus \{a\}$.

A *weak Condorcet loser* is an alternative $a \in A$ that doesn't win any head-to-head contest with some other alternative $b \in A \setminus \{a\}$. In other words:

(4.11.4) Alternative $a \in A$ is a *weak Condorcet loser* $:\Leftrightarrow$
$N[a,b] \leq N[b,a]$ for all $b \in A \setminus \{a\}$.

Suppose $\mathcal{E}$ is the set of weak Condorcet winners. Then we get:

(4.11.5) $a \in \mathcal{E} :\Leftrightarrow N[a,b] \geq N[b,a]$ for all $b \in A \setminus \{a\}$.

Suppose $\mathcal{F}$ is the set of weak Condorcet losers. Then we get:

(4.11.6) $a \in \mathcal{F} :\Leftrightarrow N[a,b] \leq N[b,a]$ for all $b \in A \setminus \{a\}$.

A frequently stated desideratum says that, when there is a weak Condorcet winner, then he should win.

When there happens to be exactly one potential winner $x \in A$ and exactly one weak Condorcet winner $y \in A$, it is obvious what the above desideratum means: Alternative x and alternative y must be the same alternative.

In other words:

(4.11.7) $\quad | \mathcal{E} | = 1$ and $| \mathcal{S} | = 1 \Rightarrow \mathcal{E} = \mathcal{S}$.

However, when there happens to be more than one potential winner or more than one weak Condorcet winner, the proper formulation for the above desideratum isn't obvious. The most intuitive formulation is:

(4.11.8) $\quad \mathcal{E} \neq \varnothing \Rightarrow \mathcal{E} = \mathcal{S}$.

Unfortunately, the following example demonstrates that (4.11.8) is incompatible with reversal symmetry:

> Suppose there are four alternatives $A = \{a,b,c,d\}$. Suppose $N^{\text{old}}[a,b] = N^{\text{old}}[b,a]$, $N^{\text{old}}[a,c] = N^{\text{old}}[c,a]$, $N^{\text{old}}[a,d] = N^{\text{old}}[d,a]$, $N^{\text{old}}[b,c] > N^{\text{old}}[c,b]$, $N^{\text{old}}[c,d] > N^{\text{old}}[d,c]$, and $N^{\text{old}}[d,b] > N^{\text{old}}[b,d]$. Then we get $\mathcal{E}^{\text{old}} = \{a\}$. With (4.11.8), we get $\mathcal{S}^{\text{old}} = \{a\}$.
>
> When the individual preferences are reversed, as defined in (4.4.1), we get $N^{\text{new}}[a,b] = N^{\text{new}}[b,a]$, $N^{\text{new}}[a,c] = N^{\text{new}}[c,a]$, $N^{\text{new}}[a,d] = N^{\text{new}}[d,a]$, $N^{\text{new}}[b,c] < N^{\text{new}}[c,b]$, $N^{\text{new}}[c,d] < N^{\text{new}}[d,c]$, and $N^{\text{new}}[d,b] < N^{\text{new}}[b,d]$. Therefore, $\mathcal{E}^{\text{new}} = \{a\}$. With (4.11.8), we get $\mathcal{S}^{\text{new}} = \{a\}$.
>
> But $\mathcal{S}^{\text{old}} = \{a\}$ and $\mathcal{S}^{\text{new}} = \{a\}$ together contradict (4.4.4).

In short: It can happen that the same alternative is the unique weak Condorcet winner in the original situation and, simultaneously, the unique weak Condorcet winner in the reversed situation. Therefore, (4.11.8) cannot be compatible with reversal symmetry.

Furthermore, the following example demonstrates that (4.11.8) is incompatible with independence of clones:

> Suppose there are only two alternatives $A^{old} = \{a,b\}$. Suppose $N[a,b] = N[b,a]$. Then we get $\mathcal{E}^{old} = \{a,b\}$. With (4.11.8), we get $\mathcal{S}^{old} = \{a,b\}$.
>
> When alternative a is replaced by alternatives a_1,a_2,a_3 such that $N[a_1,a_2] > N[a_2,a_1]$, $N[a_2,a_3] > N[a_3,a_2]$, and $N[a_3,a_1] > N[a_1,a_3]$ and such that (4.6.1) – (4.6.3) are satisfied, then we get $\mathcal{E}^{new} = \{b\}$. With (4.11.8), we get $\mathcal{S}^{new} = \{b\}$. But with (4.6.7) and $a \in \mathcal{S}^{old}$, we get $\mathcal{S}^{new} \cap \{a_1,a_2,a_3\} \neq \emptyset$. Therefore, (4.6.7) and (4.11.8) are incompatible with each other.

In short: When a weak Condorcet winner is replaced by a set of clones, as defined in (4.6.1) – (4.6.3), it is not guaranteed that at least one of these clones is a weak Condorcet winner. Therefore, (4.11.8) cannot be compatible with independence of clones.

The above examples demonstrate that, to satisfy reversal symmetry and independence of clones, we have, in some situations, to allow alternatives, which are not weak Condorcet winners, to be among the potential winners.

So the maximum, that we could ask for, is:

(4.11.9) $\mathcal{E} \subseteq \mathcal{S}$.

Formulation (4.11.9) says that every weak Condorcet winner should be a potential winner, but it makes no stipulations about those alternatives which are not weak Condorcet winners. In (4.11.9), the presumption “ $\mathcal{E} \neq \emptyset$ ” is not needed. We don’t have to write “ $\mathcal{E} \neq \emptyset \Rightarrow \mathcal{E} \subseteq \mathcal{S}$ ” because the empty set is, by definition, subset of every set.

The following proof demonstrates that the Schulze method satisfies (4.11.9) and that, therefore, (4.11.9) is compatible with reversal symmetry and independence of clones.

Claim:

If $\succ_D$ satisfies (2.1.4) and (2.1.5), then the Schulze method, as defined in section 2.2, satisfies (4.11.7) and (4.11.9).

Proof:

As (4.11.9) implies (4.11.7), it is sufficient to prove that the Schulze method satisfies (4.11.9).

Step 1:

(2.1.4) says that all ties have equivalent strengths. So without loss of generality, we can set

(4.11.10) $\forall\, x \in \mathbb{N}_0$: $(x,x) \approx_D (1,1)$.

Step 2:

Suppose $a \in A$ is a weak Condorcet winner. Then, for every $b \in A \setminus \{a\}$, the link ab is already a path from alternative a to alternative b that contains no defeat. Therefore, with (2.1.5) and (4.11.10), we get

(4.11.11) $\forall\, a \in \mathcal{E}\ \forall\, b \in A \setminus \{a\}$: $P_D[a,b] \succsim_D (N[a,b],N[b,a]) \succsim_D (1,1)$.

Step 3:

Suppose $a \in A$ is a weak Condorcet winner. Suppose $b \in A \setminus \{a\}$. Suppose the link ca is the last link in the strongest path from alternative b to alternative a. As alternative a is a weak Condorcet winner, the link ca is either a tie or a defeat. Therefore, with (2.1.5) and (4.11.10), we get

(4.11.12) $\forall\, a \in \mathcal{E}\ \forall\, b \in A \setminus \{a\}\ \exists\, c \in A \setminus \{a\}$: $P_D[b,a] \precsim_D (N[c,a],N[a,c]) \precsim_D (1,1)$.

With (4.11.11) and (4.11.12), we get

(4.11.13) $\forall\, a \in \mathcal{E}\ \forall\, b \in A \setminus \{a\}$: $P_D[a,b] \succsim_D P_D[b,a]$.

With (4.11.13), we get

(4.11.14) $a \in \mathcal{E} \Rightarrow a \in \mathcal{S}$.

With (4.11.14), we get (4.11.9). □

Another frequently stated desideratum says that a weak Condorcet loser should not win. So with (4.11.6), we get

(4.11.15) $\forall\ a \in A$: ($a \in \mathcal{F} \Rightarrow a \notin \mathcal{S}$).

However, a problem with desideratum (4.11.15) is that it can happen that alternative $a \in A$ is a weak Condorcet loser and, simultaneously, a weak Condorcet winner. In this case, alternative $a \in A$ must win according to (4.11.9) and must not win according to (4.11.15).

Example: Suppose there are only $C = 2$ alternatives $a,b \in A$. Suppose there is a pairwise tie, $N[a,b] = N[b,a]$. Then both alternatives are weak Condorcet losers and, simultaneously, weak Condorcet winners. With (4.11.9), we get $a \in \mathcal{S}$ and $b \in \mathcal{S}$. With (4.11.15), we get $a \notin \mathcal{S}$ and $b \notin \mathcal{S}$.

So the maximum, that we could ask for, is:

(4.11.16) $\forall\ a \in A$: ($a \in \mathcal{F}$ and $a \notin \mathcal{E} \Rightarrow a \notin \mathcal{S}$).

Desideratum (4.11.16) says that a weak Condorcet loser should not win, unless it is also a weak Condorcet winner. The following proof demonstrates that the Schulze method satisfies (4.11.16) and that, therefore, there is no need to weaken (4.11.16) any further.

Claim:

If $\succ_D$ satisfies (2.1.5), then the Schulze method, as defined in section 2.2, satisfies (4.11.16).

Proof:

With $a \in \mathcal{F}$, we get

(4.11.17) $\forall\ b \in A \setminus \{a\}$: $N[a,b] \leq N[b,a]$.

With $a \notin \mathcal{E}$, we get

(4.11.18) $\exists\ b \in A \setminus \{a\}$: $N[a,b] < N[b,a]$.

When we take the alternative $b \in A \setminus \{a\}$ from (4.11.18), then the link ba is already a path from alternative b to alternative a that contains no tie or defeat.

Suppose the link ac is the first link in the strongest path from alternative a to alternative b. As alternative a is a weak Condorcet loser, the link ac is either a tie or a defeat. Therefore, with (2.1.5), (4.11.17), and (4.11.18), we get

(4.11.19) $P_D[b,a] \succsim_D (N[b,a],N[a,b]) \succ_D (N[a,c],N[c,a]) \succsim_D P_D[a,b]$.

So alternative b disqualifies alternative a. So $a \notin \mathcal{S}$. □

4.12. Sequential Independence

Sequential independence says that, when alternative $a \in A$ is a winner, then there must be an alternative $d \in A \setminus \{a\}$ such that, when the used election method is applied to $A \setminus \{d\}$, then alternative a is still a winner.

The name for this criterion comes from the fact that — when the used election method satisfies this criterion and when alternative $a \in A$ is a winner and alternative $d(1) \in A \setminus \{a\}$ is an alternative such that, when the used election method is applied to $A \setminus \{d(1)\}$, then alternative a is still a winner — the same criterion can then be applied to $A \setminus \{d(1)\}$ to identify an alternative $d(2) \in A \setminus \{a,d(1)\}$ such that, when the used election method is applied to $A \setminus \{d(1),d(2)\}$, then alternative a is still a winner. When we continue applying this criterion, we get a linear order $d(1),...,d(C–1)$ of the alternatives in $A \setminus \{a\}$ such that, for every $i \in \{1,...,(C–1)\}$, alternative a is still a winner when the used election method is applied to $A \setminus \{d(1),...,d(i)\}$.

The motivation for this criterion is that an alternative $a \in A$ should be able to win only by disqualifying all the other alternatives directly or indirectly in some manner. It should not be possible that some alternatives $\varnothing \neq \{d(1),...,d(i)\} \subsetneq A$ disqualify each other in such a manner that the final winner comes from outside of $\{d(1),...,d(i)\}$. When sequential independence is satisfied, then one alternative after the other is disqualified, so that the final winner $a \in A$ can come from outside of $\{d(1),...,d(i)\}$ only when the last remaining alternative $d(j) \in \{d(1),...,d(i)\}$ is disqualified by some alternatives outside of $\{d(1),...,d(i)\}$.

Sequential independence as a criterion for single-winner election methods has been proposed by Arrow and Raynaud (1986) and generalized by Lansdowne (1996).

Definition #1:

An election method satisfies the first version of *sequential independence* if the following holds:

> Suppose alternative $a \in A$ is a unique winner when this election method is applied to A. Then there must be a (not necessarily unique) alternative $d \in A \setminus \{a\}$ such that, when this election method is applied to $A \setminus \{d\}$, then alternative a is still a unique winner.

Claim #1:

The Schulze method, as defined in section 2.2, satisfies the first version of sequential independence.

Proof of claim #1:

Suppose alternative $a \in A$ is a unique winner when this election method is applied to A. Then, according to (4.1.15), alternative a disqualifies every other alternative $b \in A \setminus \{a\}$. Therefore, we get

(4.12.1) $\quad \forall\, b \in A \setminus \{a\}: P^{\text{old}}_{D}[a,b] \succ_D P^{\text{old}}_{D}[b,a].$

Suppose $pred^{old}[a,x]$ is the predecessor of alternative $x \in A \setminus \{a\}$ in the strongest path from alternative a to alternative x, as calculated in section 2.3. Then a *leaf* is an alternative $y \in A \setminus \{a\}$ such that there is no alternative $x \in A \setminus \{a\}$ with $pred^{old}[a,x] = y$. As the strongest paths from alternative a to every other alternative $x \in A \setminus \{a\}$, as calculated by the Floyd algorithm, form an arborescence, there must be at least one leaf. Alternative d is chosen arbitrarily from these leaves.

Suppose alternative d is removed. As alternative d is a leaf, alternative d is not in the strongest path from alternative a to any other alternative $b \in A \setminus \{a,d\}$. Therefore, we get

(4.12.2) $\quad \forall\, b \in A \setminus \{a,d\}\colon P_D^{new}[a,b] \approx_D P_D^{old}[a,b].$

On the other side, when an alternative is removed, then the strengths of the strongest paths can only decrease. Therefore, we get

(4.12.3) $\quad \forall\, b \in A \setminus \{a,d\}\colon P_D^{new}[b,a] \precsim_D P_D^{old}[b,a].$

With (4.12.2), (4.12.1), and (4.12.3), we get

(4.12.4) $\quad \forall\, b \in A \setminus \{a,d\}\colon P_D^{new}[a,b] \approx_D P_D^{old}[a,b] \succ_D P_D^{old}[b,a] \succsim_D P_D^{new}[b,a]$

so that alternative a is still a unique winner when alternative d is removed. □

Definition #2:

An election method satisfies the second version of *sequential independence* if the following holds:

Suppose alternative $a \in A$ is a potential winner when this election method is applied to A. Then there must be a (not necessarily unique) alternative $d \in A \setminus \{a\}$ such that, when this election method is applied to $A \setminus \{d\}$, then alternative a is still a potential winner.

Claim #2:

The Schulze method, as defined in section 2.2, satisfies the second version of sequential independence.

Proof of claim #2:

Suppose alternative $a \in A$ is a potential winner when this election method is applied to A. Then, we get

(4.12.5) $\quad \forall\, b \in A \setminus \{a\}\colon P_D^{old}[a,b] \succsim_D P_D^{old}[b,a].$

The rest of this proof is identical to the proof of claim #1. □

4.13. *k*-Consistency

The Condorcet criterion says that, when some candidate $a \in A$ wins every head-to-head contest, then this candidate a should also be the overall winner (Condorcet, 1785).

However, many countries have a strong 3-party, 4-party or 5-party system where no single party can win a majority and where every party is willing to coalesce with every other party. In such a scenario, it seems to be rather uninteresting which candidate might win in a head-to-head contest. It is more interesting to ask whether there is some candidate who wins regardless of which candidates are nominated by the other parties.

So for example in the 3-party case with party α, party β, and party γ, it might be more interesting to ask whether there is a candidate from party α who wins every 3-way contest between himself and a candidate from party β and a candidate from party γ. If there is such a candidate, then this candidate should also be the overall winner.

More generally, if there is a $k \in \mathbb{N}$ with $k \geq 2$ such that there is an alternative $a \in A$ such that alternative a wins every k-way contest, then alternative a should also be the overall winner. This criterion is called *k-consistency*.

k-consistency as a criterion for single-winner election methods has been proposed by Heitzig (2004) and Simmons (2004). However, a similar idea had already been formulated by Saari (2001, pages 154–156). To question the relevance of the Condorcet criterion, Saari argued that it could happen that some alternative $a \in A$ wins every 2-way contest, some other alternative $b \in A \setminus \{a\}$ wins every 3-way contest, some other alternative $c \in A \setminus \{a,b\}$ wins every 4-way contest, etc., so that, with the same justification, every alternative could claim to be the overall winner. However, the fact that the Schulze method satisfies k-consistency for every $k \in \mathbb{N}$ with $k \geq 2$ means that there are election methods where Saari’s scenario is not possible, so that his criticism of the Condorcet criterion doesn’t work.

There are five different versions for *k*-consistency.

The first version addresses unique winners. This version says that, when alternative $a \in A$ is a unique winner in every *k*-way contest, then alternative *a* should also be a unique winner overall. For $k = 2$, the first version of *k*-consistency is identical to the Condorcet criterion (section 4.7).

The second version addresses potential winners. This version says that, when alternative $a \in A$ is a potential winner in every *k*-way contest, then alternative *a* should also be a potential winner overall. For $k = 2$, the second version of *k*-consistency is identical to the desideratum that weak Condorcet winners should always be potential winners (section 4.11).

The third version addresses the set of winners. This version says that, when in every *k*-way contest (that contains at least one alternative of the set $\emptyset \neq B \subsetneq A$) the winner comes from the set *B*, then the winner must also come from the set *B* when the method is applied to *A*. For $k = 2$, the third version of *k*-consistency is identical to the Smith criterion (section 4.7).

The fourth version says that, when alternative $a \in A$ is not a unique winner in any *k*-way contest, then alternative *a* should also be not a unique winner overall. For $k = 2$, the fourth version of *k*-consistency is identical to the desideratum that a weak Condorcet loser should not be a unique winner (section 4.11).

The fifth version says that, when alternative $a \in A$ is not a potential winner in any *k*-way contest, then alternative *a* should also be not a potential winner overall. For $k = 2$, the fifth version of *k*-consistency is identical to the Condorcet loser criterion (section 4.7).

4.13.1. Formulation #1

Definition:

Suppose $k \in \mathbb{N}$ with $k \geq 2$. An election method satisfies the first version of *k-consistency* if the following holds:

Suppose $C \geq k$ is the number of alternatives in A. Suppose alternative $a \in A$ is a unique winner whenever this election method is applied to some subset $\tilde{A} \subseteq A$ with $|\tilde{A}| = k$ and $a \in \tilde{A}$. Then alternative a is also a unique winner when this election method is applied to A.

Claim:

If $\succ_D$ satisfies (2.1.5), then the Schulze method, as defined in section 2.2, satisfies the first version of k-consistency for every $k \in \mathbb{N}$ with $k \geq 2$.

Proof (overview):

We will show how, when alternative $a \in A$ is not a unique winner (when this election method is applied to A), we can create, for every $k \in \mathbb{N}$ with $2 \leq k \leq C$, a subset $\tilde{A} \subseteq A$ with $|\tilde{A}| = k$ and $a \in \tilde{A}$ such that, when the Schulze method is applied to $\tilde{A}$, alternative a is not a unique winner.

Proof (details):

Suppose alternative $a \in A$ is not a unique winner when the Schulze method is applied to A. Then there must be an alternative $b \in A \setminus \{a\}$ with

(4.13.1.1) $P_D[b,a] \succsim_D P_D[a,b]$.

We set

(4.13.1.2) $(z_1,z_2) := P_D[b,a]$

to stress that this value is constant for the rest of this proof.

Suppose $c(1),...,c(n)$ is the strongest path from alternative $b \equiv c(1)$ to alternative $a \equiv c(n)$. Then we get

(4.13.1.3) $\forall\ i = 1,...,(n-1)$: $(N[c(i),c(i+1)],N[c(i+1),c(i)]) \succsim_D (z_1,z_2)$.

Especially, we get

(4.13.1.4) $(N[c(n-1),c(n)],N[c(n),c(n-1)]) \succsim_D (z_1,z_2)$.

When there is more than one path from alternative b to alternative a of strength (z_1,z_2) then, without loss of generality, we take the shortest of these paths (in terms of its number of links). Therefore, we get

(4.13.1.5) $\forall\ i,j \in \{1,...,n\}$ with $j - i \geq 2$: $(N[c(i),c(j)],N[c(j),c(i)]) \prec_D (z_1,z_2)$.

Otherwise, if there was a link $c(i),c(j)$ with $(N[c(i),c(j)],N[c(j),c(i)]) \succsim_D (z_1,z_2)$ and $j - i \geq 2$, then we could find a shorter path of strength (z_1,z_2) by omitting the alternatives $c(i+1),...,c(j-1)$. This would be a contradiction to the presumption that $c(1),...,c(n)$ is the shortest path of strength (z_1,z_2).

With (2.1.5), we get that every path that contains no defeat is always stronger than every path that contains a defeat.

It is easy to prove that, for every pair of alternatives $x,y \in A$, there is a path from alternative x to alternative y that contains no defeat or a path from alternative y to alternative x that contains no defeat. To prove this, we only have to consider the links xy and yx because the link xy is already a path from alternative x to alternative y and the link yx is already a path from alternative y to alternative x. If $N[x,y] > N[y,x]$, then the link xy is a path from alternative x to alternative y that contains no defeat. If $N[x,y] < N[y,x]$, then the link yx is a path from alternative y to alternative x that contains no defeat. If $N[x,y] = N[y,x]$, then the link xy is a path from alternative x to alternative y that contains no defeat and the link yx is a path from alternative y to alternative x that contains no defeat.

With (4.13.1.1) and the above considerations, we get that the path $c(1),...,c(n)$ contains no defeat. {Otherwise: Suppose the path $c(1),...,c(n)$ contains a defeat. Then [as, for every pair of alternatives $x,y \in A$, there is a path from alternative x to alternative y that contains no defeat or a path from alternative y to alternative x that contains no defeat] there must be a path $d(1),...,d(r)$ from alternative b to alternative a that contains no defeat or a path $e(1),...,e(s)$ from alternative a to alternative b that contains no defeat. If there is a path $d(1),...,d(r)$ from alternative b to alternative a that contains no defeat then, according to (2.1.5), this path is stronger than the path $c(1),...,c(n)$ that contains a defeat; this is a contradiction to the presumption that the path $c(1),...,c(n)$ is the strongest path from alternative b to alternative a. If there is no path from alternative b to alternative a that contains no defeat, but a path $e(1),...,e(s)$ from alternative a to alternative b that contains no defeat then, according to (2.1.5), this path is stronger than the path $c(1),...,c(n)$ that contains a defeat; this is a contradiction to (4.13.1.1).} Especially, the link $c(n-1),c(n)$ is not a defeat. Therefore, we get

(4.13.1.6) $\quad \forall\ i = 1,...,(n-1)$: $N[c(i),c(i+1)] \geq N[c(i+1),c(i)]$.

Especially, we get

(4.13.1.7) $\quad N[c(n-1),c(n)] \geq N[c(n),c(n-1)]$.

With (2.1.5) and (4.13.1.7), we get

(4.13.1.8) $\quad (N[c(n-1),c(n)],N[c(n),c(n-1)]) \gtrsim_D (N[c(n),c(n-1)],N[c(n-1),c(n)])$.

With the above considerations, we can now show how the subset $\tilde{A} \subseteq A$ can be chosen.

Case #1: $k = 2$.

When $\succ_D$ satisfies (2.1.5), then the first version of 2-consistency, applied to the Schulze method, means that the Schulze method should satisfy the Condorcet criterion. However, it has already been proven in section 4.7 that the Schulze method satisfies the Condorcet criterion when $\succ_D$ satisfies (2.1.5).

Case #2: $3 \leq k < n$.

Here, we choose $\tilde{A} := \{c(1),...,c(k–2),c(n–1),c(n)\}$.

When the Schulze method is applied to $\tilde{A}$, then there is a path from $c(n–1)$ to $c(n)$ of at least $(N[c(n–1),c(n)],N[c(n),c(n–1)]) \gtrsim_D (z_1,z_2)$ because, according to (4.13.1.4), already the link $c(n–1),c(n)$ is a path from $c(n–1)$ to $c(n)$ of this strength.

On the other side, there cannot be a path in $\tilde{A}$ from $c(n)$ to $c(n–1)$ of more than $(N[c(n–1),c(n)],N[c(n),c(n–1)])$ because, according to (4.13.1.5), every link from $c(1)$, ..., $c(k–2)$ to $c(n–1)$ is weaker than (z_1,z_2) and, according to (4.13.1.8), the link $c(n),c(n–1)$ is not stronger than $(N[c(n–1),c(n)],N[c(n),c(n–1)])$.

Therefore, alternative $c(n)$ cannot disqualify alternative $c(n–1)$. So either alternative $c(n–1)$ is also a potential winner or, according to (4.1.14), alternative $c(n–1)$ must be disqualified by some other potential winner. In both cases, alternative $c(n)$ is not a unique winner.

Case #3: $k \geq n$.

Here, $\tilde{A}$ consists of the alternatives $c(1),...,c(n)$ and $k–n$ additional alternatives from A.

As $\{c(1),...,c(n)\} \subseteq \tilde{A}$, there is a path in $\tilde{A}$ from alternative $c(1)$ to alternative $c(n)$ of strength (z_1,z_2). On the other side, we get, with (4.13.1.1), that there cannot be a path in $\tilde{A}$ from alternative $c(n)$ to alternative $c(1)$ of more than (z_1,z_2) because, when alternatives are removed from A, then the strength of the strongest path from alternative $c(n)$ to alternative $c(1)$ can only decrease.

Therefore, alternative $c(n)$ cannot disqualify alternative $c(1)$. So either alternative $c(1)$ is also a potential winner or, according to (4.1.14), alternative $c(1)$ must be disqualified by some other potential winner. In both cases, alternative $c(n)$ is not a unique winner. □

4.13.2. Formulation #2

Definition:

Suppose $k \in \mathbb{N}$ with $k \geq 2$. An election method satisfies the second version of *k-consistency* if the following holds:

> Suppose $C \geq k$ is the number of alternatives in A. Suppose alternative $a \in A$ is a potential winner whenever this election method is applied to some subset $\tilde{A} \subseteq A$ with $|\tilde{A}| = k$ and $a \in \tilde{A}$. Then alternative a is also a potential winner when this election method is applied to A.

Claim:

If $\succ_D$ satisfies (2.1.4) and (2.1.5), then the Schulze method, as defined in section 2.2, satisfies the second version of k-consistency for every $k \in \mathbb{N}$ with $k \geq 2$.

Proof (overview):

We will show how, when alternative $a \in A$ is not a potential winner (when this election method is applied to A), we can create, for every $k \in \mathbb{N}$ with $2 \leq k \leq C$, a subset $\tilde{A} \subseteq A$ with $|\tilde{A}| = k$ and $a \in \tilde{A}$ such that, when the Schulze method is applied to $\tilde{A}$, alternative a is not a potential winner.

Proof (details):

Suppose alternative $a \in A$ is not a potential winner when the Schulze method is applied to A. Then there must be an alternative $b \in A \setminus \{a\}$ with

(4.13.2.1) $\quad P_D[b,a] \succ_D P_D[a,b]$.

We set

(4.13.2.2) $\quad (z_1,z_2) := P_D[b,a]$

to stress that this value is constant for the rest of this proof.

Suppose $c(1),...,c(n)$ is the strongest path from alternative $b \equiv c(1)$ to alternative $a \equiv c(n)$. Then we get

(4.13.2.3) $\quad \forall\, i = 1,...,(n-1)\colon (N[c(i),c(i+1)],N[c(i+1),c(i)]) \succsim_D (z_1,z_2)$.

Especially, we get

(4.13.2.4) $\quad (N[c(n-1),c(n)],N[c(n),c(n-1)]) \succsim_D (z_1,z_2)$.

With the same arguments as for (4.13.1.5), we get

(4.13.2.5) $\quad \forall\, i,j \in \{1,...,n\}$ with $j - i \geq 2\colon (N[c(i),c(j)],N[c(j),c(i)]) \prec_D (z_1,z_2)$.

With (2.1.4) and (2.1.5), we get that every path that contains no defeat or tie is always stronger than every path that contains a defeat or tie.

It is easy to prove that the path $c(1),...,c(n)$ contains no defeat or tie. Therefore, we get

(4.13.2.6) $\forall\ i = 1,...,(n-1)$: $N[c(i),c(i+1)] > N[c(i+1),c(i)]$.

Especially, we get

(4.13.2.7) $N[c(n-1),c(n)] > N[c(n),c(n-1)]$.

With (2.1.5) and (4.13.2.7), we get

(4.13.2.8) $(N[c(n-1),c(n)],N[c(n),c(n-1)]) \succ_D (N[c(n),c(n-1)],N[c(n-1),c(n)])$.

Proof for (4.13.2.6):

It has already been shown in the proof in section 4.13.1 that, when $\succ_D$ satisfies (2.1.5), then the path $c(1),...,c(n)$ contains no defeat. So it remains to be proven that the path $c(1),...,c(n)$ contains no tie.

To prove that the path $c(1),...,c(n)$ contains no tie, we presume that (2.1.4), (2.1.5), and (4.13.2.1) are satisfied and that the path $c(1),...,c(n)$ contains a tie and then we will show that this leads to a contradiction.

(2.1.4) says that all ties have equivalent strengths. (2.1.5) says that every win is stronger than every tie. So when the path $c(1),...,c(n)$ contains no defeat, but at least one tie then, without loss of generality, we can set

(4.13.2.9) $P_D[b,a] \approx_D (1,1)$.

To get to a contradiction, it is sufficient to consider the link ab.

Case #A: If the link ab is a win (i.e. $N[a,b] > N[b,a]$) or a tie (i.e. $N[a,b] = N[b,a]$), then this link is already a path from alternative a to alternative b that contains no defeat. Therefore, with (2.1.4), (2.1.5), and (4.13.2.9), we get $P_D[a,b] \succsim_D (N[a,b],N[b,a]) \succsim_D (1,1) \approx_D P_D[b,a]$. But this is a contradiction to (4.13.2.1).

Case #B: If the link ab is a defeat (i.e. $N[a,b] < N[b,a]$), then the link ba is a path from alternative b to alternative a that contains no defeat or tie. But then, according to (2.1.5), the link ba is stronger than the path $c(1),...,c(n)$ that contains a tie. But this is a contradiction to the presumption that the path $c(1),...,c(n)$ is the strongest path from alternative b to alternative a.

With the above considerations, we can now show how the subset $\tilde{A} \subseteq A$ can be chosen.

Case #1: $k = 2$.

When $\succ_D$ satisfies (2.1.4) and (2.1.5), then the second version of 2-consistency, applied to the Schulze method, means that the Schulze method should satisfy the desideratum that a weak Condorcet winner is always a potential winner. However, it has already been proven in section 4.11 that the Schulze method satisfies this desideratum when $\succ_D$ satisfies (2.1.4) and (2.1.5).

Case #2: $3 \leq k < n$.

Here, we choose $\tilde{A} := \{c(1),...,c(k-2),c(n-1),c(n)\}$.

When the Schulze method is applied to $\tilde{A}$, then there is a path from $c(n-1)$ to $c(n)$ of at least $(N[c(n-1),c(n)],N[c(n),c(n-1)]) \gtrsim_D (z_1,z_2)$ because, according to (4.13.2.4), already the link $c(n-1),c(n)$ is a path from $c(n-1)$ to $c(n)$ of this strength.

On the other side, there cannot be a path in $\tilde{A}$ from $c(n)$ to $c(n-1)$ of at least $(N[c(n-1),c(n)],N[c(n),c(n-1)])$ because, according to (4.13.2.5), every link from $c(1)$, ..., $c(k-2)$ to $c(n-1)$ is weaker than (z_1,z_2) and, according to (4.13.2.8), the link $c(n),c(n-1)$ is weaker than $(N[c(n-1),c(n)],N[c(n),c(n-1)])$.

Therefore, alternative $c(n-1)$ disqualifies alternative $c(n)$, so that alternative $c(n)$ is not a potential winner.

Case #3: $k \geq n$.

Here, $\tilde{A}$ consists of the alternatives $c(1),...,c(n)$ and $k-n$ additional alternatives from A.

As $\{c(1),...,c(n)\} \subseteq \tilde{A}$, there is a path in $\tilde{A}$ from alternative $c(1)$ to alternative $c(n)$ of strength (z_1,z_2). On the other side, we get, with (4.13.2.1), that there cannot be a path in $\tilde{A}$ from alternative $c(n)$ to alternative $c(1)$ of at least (z_1,z_2) because, when alternatives are removed from A, then the strength of the strongest path from alternative $c(n)$ to alternative $c(1)$ can only decrease.

Therefore, alternative $c(1)$ disqualifies alternative $c(n)$, so that alternative $c(n)$ is not a potential winner. □

4.13.3. Formulation #3

Definition:

Suppose $k \in \mathbb{N}$ with $k \geq 2$. An election method satisfies the third version of *k-consistency* if the following holds:

Suppose $C \geq k$ is the number of alternatives in A. Suppose $\mathcal{S}|_{\tilde{A}}$ is the set of potential winners when this election method is applied to $\varnothing \neq \tilde{A} \subseteq A$. Suppose $\varnothing \neq B \subsetneq A$. Suppose $\mathcal{S}|_{\tilde{A}} \subseteq B$ whenever this election method is applied to some subset $\tilde{A} \subseteq A$ with $|\tilde{A}| = k$ and $B \cap \tilde{A} \neq \varnothing$. Then we must also get $\mathcal{S}|_A \subseteq B$. In short:

$\forall \varnothing \neq B \subsetneq A$: (($\forall \tilde{A} \subseteq A$ with $|\tilde{A}| = k$ and $B \cap \tilde{A} \neq \varnothing$: $\mathcal{S}|_{\tilde{A}} \subseteq B$) $\Rightarrow$ ($\mathcal{S}|_A \subseteq B$)).

Claim:

If $\succ_D$ satisfies (2.1.5), then the Schulze method, as defined in section 2.2, satisfies the third version of k-consistency for every $k \in \mathbb{N}$ with $k \geq 2$.

Proof (overview):

We will show how, when $\mathcal{S}|_A \nsubseteq B$, we can create, for every $k \in \mathbb{N}$ with $2 \leq k \leq C$, a subset $\tilde{A} \subseteq A$ with $|\tilde{A}| = k$ and $B \cap \tilde{A} \neq \varnothing$ such that, when the Schulze method is applied to $\tilde{A}$, we get $\mathcal{S}|_{\tilde{A}} \nsubseteq B$.

Proof (details):

Suppose $r := |B|$ is the number of alternatives in B. With $\varnothing \neq B \subsetneq A$, we get: $0 < r < C$.

Suppose $\mathcal{S}|_A \nsubseteq B$. Then there must be an alternative $b \in A$ with $b \in \mathcal{S}|_A$ and $b \notin B$. With $b \in \mathcal{S}|_A$ we get

(4.13.3.1) $\forall a \in A \setminus \{b\}: P_D[b,a] \succsim_D P_D[a,b]$.

Case #1: $k = 2$.

When $\succ_D$ satisfies (2.1.5), then the third version of 2-consistency, applied to the Schulze method, means that the Schulze method should satisfy the Smith criterion. However, it has already been proven in section 4.7 that the Schulze method satisfies the Smith criterion when $\succ_D$ satisfies (2.1.5).

Case #2: $k > C - r$.

In section 4.12, we have proven that, when alternative $b \in A$ is a potential winner, then there is a linear order $d(1),...,d(C-1)$ of the alternatives in $A \setminus \{b\}$, such that, when the Schulze method is applied to $A \setminus \{d(1),...,d(C-k)\}$, then alternative b is still a potential winner.

As $k > C - r$, every set $\tilde{A} \subseteq A$ with $|\tilde{A}| = k$ contains at least $k + r - C \geq 1$ alternatives of B. Therefore, we get $B \cap \tilde{A} \neq \emptyset$ for every set $\tilde{A} \subseteq A$ with $|\tilde{A}| = k$. Therefore, we can choose $\tilde{A} := A \setminus \{d(1),...,d(C-k)\}$.

Case #3: $3 \leq k \leq C - r$.

We take some $b \in A$ with $b \in \mathcal{S}|_A$ and $b \notin B$. We sort the alternatives $\{a(1),...,a(C-1)\}$ in $A \setminus \{b\}$ such that

$\forall\, i,j \in \mathbb{N}$ with $1 \leq i < C$ and $1 \leq j < C$: ($pred[b,a(j)] = a(i) \Rightarrow i < j$).

Suppose $y \in \mathbb{N}$ with $1 \leq y < C$ is the smallest number with $a(y) \in B$. Then we get $a(x) \notin B$ for all $x \in \mathbb{N}$ with $1 \leq x < y$. Furthermore, when $d(1),...,d(m)$ is the strongest path from alternative $b \equiv d(1)$ to alternative $a(y) \equiv d(m)$ then, with the definition for $pred[i,j]$ and with the definition for the order of $\{a(1),...,a(C-1)\}$, we get $\{d(1),...,d(m-1)\} \subseteq \{b,a(1),...,a(y-1)\} \subseteq A \setminus B$.

We set

(4.13.3.2) $\quad (z_1,z_2) := P_D[b,a(y)]$

to stress that this value is constant for the rest of this proof.

We now shorten the path $d(1),...,d(m)$ by removing possible short cuts. So when there is a link $d(i),d(j)$ with $(N[d(i),d(j)],N[d(j),d(i)]) \succsim_D (z_1,z_2)$ and $j - i \geq 2$, we remove the alternatives $d(i+1),...,d(j-1)$ from this path. We continue removing possible short cuts, until the resulting path contains no short cuts anymore. The resulting path will be called $c(1),...,c(n)$.

We get $c(i) \notin B$ for all $i \in \mathbb{N}$ with $1 \leq i < n$, because we have already established $d(i) \notin B$ for all $i \in \mathbb{N}$ with $1 \leq i < m$ and because, when we shortened the path $d(1),...,d(m)$, we only removed and didn’t add alternatives.

With the same arguments as for (4.13.1.3) – (4.13.1.8), we get (4.13.3.3) – (4.13.3.8):

(4.13.3.3) $\quad \forall\, i = 1,...,(n-1)$: $(N[c(i),c(i+1)],N[c(i+1),c(i)]) \succsim_D (z_1,z_2)$.

(4.13.3.4) $\quad (N[c(n-1),c(n)],N[c(n),c(n-1)]) \succsim_D (z_1,z_2)$.

(4.13.3.5) $\quad \forall\, i,j \in \{1,...,n\}$ with $j - i \geq 2$: $(N[c(i),c(j)],N[c(j),c(i)]) \prec_D (z_1,z_2)$.

(4.13.3.6) $\quad \forall\, i = 1,...,(n-1)$: $N[c(i),c(i+1)] \geq N[c(i+1),c(i)]$.

(4.13.3.7) $N[c(n-1),c(n)] \geq N[c(n),c(n-1)]$.

(4.13.3.8) $(N[c(n-1),c(n)],N[c(n),c(n-1)]) \succsim_D (N[c(n),c(n-1)],N[c(n-1),c(n)])$.

With the above considerations, we can now show how the subset $\tilde{A} \subseteq A$ can be chosen.

<u>Case #3a:</u> $3 \leq k < n$.

Here, we choose $\tilde{A} := \{c(1),...,c(k-2),c(n-1),c(n)\}$.

When the Schulze method is applied to $\tilde{A}$, then there is a path from $c(n-1)$ to $c(n)$ of at least $(N[c(n-1),c(n)],N[c(n),c(n-1)]) \succsim_D (z_1,z_2)$ because, according to (4.13.3.4), already the link $c(n-1),c(n)$ is a path from $c(n-1)$ to $c(n)$ of this strength.

On the other side, there cannot be a path in $\tilde{A}$ from $c(n)$ to $c(n-1)$ of more than $(N[c(n-1),c(n)],N[c(n),c(n-1)])$ because, according to (4.13.3.5), every link from $c(1)$, ..., $c(k-2)$ to $c(n-1)$ is weaker than (z_1,z_2) and, according to (4.13.3.8), the link $c(n),c(n-1)$ is not stronger than $(N[c(n-1),c(n)],N[c(n),c(n-1)])$.

Therefore, alternative $c(n)$ cannot disqualify alternative $c(n-1)$. So either alternative $c(n-1)$ is also a potential winner or, according to (4.1.14), alternative $c(n-1)$ must be disqualified by some other potential winner in $\tilde{A}$. As $c(i) \notin B$ for all $i \in \mathbb{N}$ with $1 \leq i < n$, this potential winner comes from outside B.

<u>Case #3b:</u> $n \leq k \leq C - r$.

Here, $\tilde{A}$ consists of the alternatives $c(1),...,c(n)$ and $k-n$ additional alternatives from $A \setminus B$.

As $\{c(1),...,c(n)\} \subseteq \tilde{A}$, there is a path in $\tilde{A}$ from alternative $c(1)$ to alternative $c(n)$ of strength (z_1,z_2). On the other side, we get, with (4.13.3.1), that there cannot be a path in $\tilde{A}$ from alternative $c(n)$ to alternative $c(1)$ of more than (z_1,z_2) because, when alternatives are removed from A, then the strength of the strongest path from alternative $c(n)$ to alternative $c(1)$ can only decrease.

Therefore, alternative $c(n)$ cannot disqualify alternative $c(1)$. So either alternative $c(1)$ is also a potential winner or, according to (4.1.14), alternative $c(1)$ must be disqualified by some other potential winner in $\tilde{A}$. As $e \notin B$ for all $e \in \tilde{A} \setminus \{c(n)\}$, this potential winner comes from outside B. □

4.13.4. Formulation #4

Definition:

Suppose $k \in \mathbb{N}$ with $k \geq 2$. An election method satisfies the fourth version of *k-consistency* if the following holds:

> Suppose $C \geq k$ is the number of alternatives in A. Suppose alternative $a \in A$ is not a unique winner whenever this election method is applied to some subset $\tilde{A} \subseteq A$ with $|\tilde{A}| = k$ and $a \in \tilde{A}$. Then alternative a is also not a unique winner when this election method is applied to A.

Claim:

The Schulze method, as defined in section 2.2, satisfies the fourth version of k-consistency for every $k \in \mathbb{N}$ with $k \geq 2$.

Remark:

Presumptions (2.1.4) and (2.1.5) are not needed in the following proof. However, only when $\succ_D$ satisfies (2.1.4) and (2.1.5), the fourth version of k-consistency with $k = 2$ is identical to the desideratum that a weak Condorcet loser should not be a unique winner.

Proof (overview):

We will show how, when alternative $a \in A$ is a unique winner (when this election method is applied to A), we can create, for every $k \in \mathbb{N}$ with $2 \leq k \leq C$, a subset $\tilde{A} \subseteq A$ with $|\tilde{A}| = k$ and $a \in \tilde{A}$ such that, when the Schulze method is applied to $\tilde{A}$, alternative a is a unique winner.

Proof (details):

In section 4.12, we have proven that, when alternative $a \in A$ is a unique winner, then there is a linear order $d(1),...,d(C-1)$ of the alternatives in $A \setminus \{a\}$ such that, for every $i \in \{1,...,(C-1)\}$, alternative a is still a unique winner when the Schulze method is applied to $A \setminus \{d(1),...,d(i)\}$.

Therefore, for $k \in \mathbb{N}$ with $2 \leq k \leq C$, we can simply choose $\tilde{A} := A \setminus \{d(1),...,d(C-k)\}$. □

4.13.5. Formulation #5

Definition:

Suppose $k \in \mathbb{N}$ with $k \geq 2$. An election method satisfies the fifth version of *k-consistency* if the following holds:

> Suppose $C \geq k$ is the number of alternatives in A. Suppose alternative $a \in A$ is not a potential winner whenever this election method is applied to some subset $\tilde{A} \subseteq A$ with $|\tilde{A}| = k$ and $a \in \tilde{A}$. Then alternative a is also not a potential winner when this election method is applied to A.

Claim:

The Schulze method, as defined in section 2.2, satisfies the fifth version of k-consistency for every $k \in \mathbb{N}$ with $k \geq 2$.

Remark:

Presumption (2.1.5) is not needed in the following proof. However, only when $\succ_D$ satisfies (2.1.5), the fifth version of k-consistency with $k = 2$ is identical to the Condorcet loser criterion.

Proof (overview):

We will show how, when alternative $a \in A$ is a potential winner (when this election method is applied to A), we can create, for every $k \in \mathbb{N}$ with $2 \leq k \leq C$, a subset $\tilde{A} \subseteq A$ with $|\tilde{A}| = k$ and $a \in \tilde{A}$ such that, when the Schulze method is applied to $\tilde{A}$, alternative a is a potential winner.

Proof (details):

In section 4.12, we have proven that, when alternative $a \in A$ is a potential winner, then there is a linear order $d(1),...,d(C\text{–}1)$ of the alternatives in $A \setminus \{a\}$ such that, for every $i \in \{1,...,(C\text{–}1)\}$, alternative a is still a potential winner when the Schulze method is applied to $A \setminus \{d(1),...,d(i)\}$.

Therefore, for $k \in \mathbb{N}$ with $2 \leq k \leq C$, we can simply choose $\tilde{A} := A \setminus \{d(1),...,d(C\text{–}k)\}$. □

5. Tie-Breaking

It can happen that the weakest link in the strongest path from alternative *a* to alternative *b* and the weakest link in the strongest path from alternative *b* to alternative *a* are the same link, say *cd*. In this case, the Schulze method is indifferent between alternative *a* and alternative *b*, i.e. $ab \notin \mathcal{O}$ and $ba \notin \mathcal{O}$. See sections 3.3, 3.8, 3.9, and 4.2.

In this section, we recommend that, to resolve this indifference, the link *cd* should be declared *forbidden* and the strongest paths from alternative *a* to alternative *b* and from alternative *b* to alternative *a*, that don’t contain *forbidden* links, should be calculated. Either this indifference is now resolved or, again, the weakest link in the strongest path from alternative *a* to alternative *b* and the weakest link in the strongest path from alternative *b* to alternative *a* are the same link, say *ef*. In the latter case, the link *ef* is declared *forbidden* and the strongest paths that don’t contain *forbidden* links are calculated. This procedure is repeated until this indifference is resolved.

The resulting Schulze relation will be called $\mathcal{O}_{final}$. The resulting set of winners will be called $\mathcal{S}_{final}$. The precise definitions for $\mathcal{O}_{final}$ and $\mathcal{S}_{final}$ will be given in (5.1.2) and (5.1.3).

In example 3 (section 3.3), the link *cd* is the weakest link in the strongest path from alternative *a* to alternative *b* and the weakest link in the strongest path from alternative *b* to alternative *a*. Therefore, the link *cd* is declared *forbidden*. The strongest path from alternative *a* to alternative *b*, that doesn’t contain *forbidden* links, is *a*,(33,30),*b*. The strongest path from alternative *b* to alternative *a*, that doesn’t contain *forbidden* links, is *b*,(30,33),*a*. Therefore, we get $ab \in \mathcal{O}_{final}$.

5.1. Calculating a Complete Ranking Using a Tie-Breaking Ranking of the Links

Suppose $\mathcal{LO}_{A \times A}$ is the set of linear orders on $A \times A$. Then a *Tie-Breaking Ranking of the Links* (TBRL) is a linear order $\sigma \in \mathcal{LO}_{A \times A}$ with the following property:

$$(5.1.1) \qquad (N[i,j],N[j,i]) \succ_D (N[m,n],N[n,m]) \Rightarrow ij \succ_\sigma mn.$$

Suppose $\sigma \in \mathcal{LO}_{A \times A}$ is a linear order on $A \times A$ with property (5.1.1). Then we calculate $\mathcal{O}_{final}(\sigma)$ and $\mathcal{S}_{final}(\sigma)$ as described in stages 1–4:

Stage 1 (initialization):

```
for i : = 1 to C
begin
    for j : = 1 to C
    begin
        if ( i ≠ j ) then
        begin
            Pσ[i,j] : = ij
        end
    end
end
```

Stage 2 (calculation of the strengths of the strongest paths):

```
for i : = 1 to C
begin
    for j : = 1 to C
    begin
        if ( i ≠ j ) then
        begin
            for k : = 1 to C
            begin
                if ( i ≠ k ) then
                begin
                    if ( j ≠ k ) then
                    begin
                        if ( Pσ[j,k] ≺σ minσ { Pσ[j,i], Pσ[i,k] } ) then
                        begin
                            Pσ[j,k] : = minσ { Pσ[j,i], Pσ[i,k] }
                        end
                    end
                end
            end
        end
    end
end
```

Stage 3 (calculation of the binary relation $\mathcal{O}$ and the set of potential winners):

```
𝒪final(σ) : = ∅
𝒮final(σ) : = A
for i : = 1 to C
begin
    for j : = 1 to C
    begin
        if ( i ≠ j ) then
        begin
            if ( Pσ[j,i] ≻σ Pσ[i,j] ) then
            begin
                𝒪final(σ) : = 𝒪final(σ) + {ji}
                𝒮final(σ) : = 𝒮final(σ) \ {i}
            end
        end
    end
end
```

Stage 4 (tie-breaking):

```
xy : = minσ { ij | i,j ∈ {1,...,C}, i ≠ j }
for m : = 1 to C–1
begin
    for n : = m+1 to C
    begin
        if ( Pσ[m,n] ≈σ Pσ[n,m] ) then
        begin
            for i : = 1 to C
            begin
                for j : = 1 to C
                begin
                    if ( i ≠ j ) then
                    begin
                        forbidden[i,j] : = false
                        Qσ[i,j] : = Pσ[i,j]
                    end
                end
            end
            bool1 : = false
            while ( bool1 = false )
            begin
                for i : = 1 to C
                begin
                    for j : = 1 to C
                    begin
                        if ( i ≠ j ) then
                        begin
                            if ( Qσ[m,n] ≈σ ij ) then
                            begin
                                forbidden[i,j] : = true
                            end
                        end
                    end
                end
                for i : = 1 to C
                begin
                    for j : = 1 to C
                    begin
                        if ( i ≠ j ) then
                        begin
                            if ( forbidden[i,j] = true ) then
                            begin
                                Qσ[i,j] : = xy
                            end
                            else
                            begin
                                Qσ[i,j] : = ij
                            end
                        end
                    end
                end
```

```
100                 for i : = 1 to C
101                 begin
102                     for j : = 1 to C
103                     begin
104                         if ( i ≠ j ) then
105                         begin
106                             for k : = 1 to C
107                             begin
108                                 if ( i ≠ k ) then
109                                 begin
110                                     if ( j ≠ k ) then
111                                     begin
112                                         if ( Qσ[j,k] ≺σ minσ { Qσ[j,i], Qσ[i,k] } ) then
113                                         begin
114                                             Qσ[j,k] : = minσ { Qσ[j,i], Qσ[i,k] }
115                                         end
116                                     end
117                                 end
118                             end
119                         end
120                     end
121                 end
122                 if ( Qσ[m,n] ≻σ Qσ[n,m] ) then
123                 begin
124                     O_final(σ) : = O_final(σ) + {mn}
125                     S_final(σ) : = S_final(σ) \ {n}
126                     bool1 : = true
127                 end
128                 else
129                 if ( Qσ[m,n] ≺σ Qσ[n,m] ) then
130                 begin
131                     O_final(σ) : = O_final(σ) + {nm}
132                     S_final(σ) : = S_final(σ) \ {m}
133                     bool1 : = true
134                 end
135             end
136         end
137     end
138 end
```

For each pair of alternatives $m,n \in A$, we check whether $P_\sigma[m,n] \approx_\sigma P_\sigma[n,m]$ (lines 50–55). In this case, the link ij with $P_\sigma[m,n] \approx_\sigma ij$ is declared *forbidden* (lines 70–82) and the strongest paths, that don't contain *forbidden* links, are calculated (lines 83–121). This procedure is repeated (lines 67–68) until this indifference is resolved (lines 122–134).

We define

(5.1.2) $\quad \mathcal{O}_{final} := \cap \{ \mathcal{O}_{final}(\sigma) \mid \sigma \in \mathcal{LO}_{A \times A} \text{ with (5.1.1)} \}.$

(5.1.3) $\quad \mathcal{S}_{final} := \cup \{ \mathcal{S}_{final}(\sigma) \mid \sigma \in \mathcal{LO}_{A \times A} \text{ with (5.1.1)} \}.$

5.2. Calculating a Tie-Breaking Ranking of the Candidates and a Tie-Breaking Ranking of the Links

The Schulze relation $\mathcal{O}$, as defined in (2.2.1), is only a strict partial order. However, sometimes, a linear order is needed. In this section, we will show how the Schulze relation $\mathcal{O}$ can be completed to a linear order without having to sacrifice any of the desired criteria.

Step 1:

A *Tie-Breaking Ranking of the Links* (TBRL), a linear order $\succ_\sigma$ on $A \times A$, and a *Tie-Breaking Ranking of the Candidates* (TBRC), a linear order $\succ_\mu$ on A, are calculated as follows:

a) In the beginning:

- $\forall\ (i,j),(m,n) \in A \times A$: $(N[i,j],N[j,i]) \succ_D (N[m,n],N[n,m]) \Rightarrow ij \succ_\sigma mn$.
- $\forall\ (i,j),(m,n) \in A \times A$: $(N[i,j],N[j,i]) \approx_D (N[m,n],N[n,m]) \Rightarrow ij \approx_\sigma mn$.
- $\forall\ i,j \in A$: $i \approx_\mu j$.

b) Pick a random ballot $v \in V$ and use its rankings. That means:

- $\forall\ (i,j),(m,n) \in A \times A$: If $ij \approx_\sigma mn$ and

  (5.2.1) $$((i \succ_v j) \wedge (m \precsim_v n)) \vee ((i \succsim_v j) \wedge (m \prec_v n))$$

  then replace “ $ij \approx_\sigma mn$ ” by “ $ij \succ_\sigma mn$ ”.

- $\forall\ i,j \in A$: If $i \approx_\mu j$ and $i \succ_v j$, then replace “ $i \approx_\mu j$ ” by “ $i \succ_\mu j$ ”.

When the bylaws require that the chairperson decides in the case of a tie, then, for the calculations of the TBRL and the TBRC, the ballot of the chairperson has to be chosen first.

c) Continue picking ballots randomly from those that have not yet been picked and use their rankings.

d) If you go through all ballots and there are still alternatives $i,j \in A$ with $i \approx_\mu j$, then proceed as follows:

d1) Pick a random alternative k and complete the TBRC in its favor. (That means: For all alternatives $l \in A \setminus \{k\}$ with $k \approx_\mu l$: Replace “ $k \approx_\mu l$ ” by “ $k \succ_\mu l$ ”.)

d2) Continue picking alternatives randomly from those that have not yet been picked and complete the TBRC in their favor.

Step 2:

Suppose there are still $(i,j),(m,n) \in A \times A$ with $ij \approx_\sigma mn$, then proceed as follows:

Variant 1: When at least one of the following conditions is satisfied, then replace “ $ij \approx_\sigma mn$ ” by “ $ij \succ_\sigma mn$ ”:

(5.2.2a) $i \succ_\mu j$ and $n \succ_\mu m$.
(5.2.3a) $i \succ_\mu j$ and $m \succ_\mu n$ and $i \succ_\mu m$.
(5.2.4a) $j \succ_\mu i$ and $n \succ_\mu m$ and $n \succ_\mu j$.
(5.2.5a) $i \equiv m$ and $n \succ_\mu j$.
(5.2.6a) $j \equiv n$ and $i \succ_\mu m$.

Variant 2: When at least one of the following conditions is satisfied, then replace “ $ij \approx_\sigma mn$ ” by “ $ij \succ_\sigma mn$ ”:

(5.2.2b) $i \succ_\mu j$ and $n \succ_\mu m$.
(5.2.3b) $i \succ_\mu j$ and $m \succ_\mu n$ and $n \succ_\mu j$.
(5.2.4b) $j \succ_\mu i$ and $n \succ_\mu m$ and $i \succ_\mu m$.
(5.2.5b) $i \equiv m$ and $n \succ_\mu j$.
(5.2.6b) $j \equiv n$ and $i \succ_\mu m$.

(5.2.2a) – (5.2.6a) and (5.2.2b) – (5.2.6b) are chosen in such a manner that e.g. when the TBRC $\succ_\mu$ is *abcdefgh* then links of otherwise equivalent strengths are sorted *ah*, *ag*, *af*, *ae*, *ad*, *ac*, *ab*, *bh*, *bg*, *bf*, *be*, *bd*, *bc*, *ch*, *cg*, *cf*, *ce*, *cd*, *dh*, *dg*, *df*, *de*, *eh*, *eg*, *ef*, *fh*, *fg*, *gh*, *hg*, *gf*, *hf*, *fe*, *ge*, *he*, *ed*, *fd*, *gd*, *hd*, *dc*, *ec*, *fc*, *gc*, *hc*, *cb*, *db*, *eb*, *fb*, *gb*, *hb*, *ba*, *ca*, *da*, *ea*, *fa*, *ga*, *ha* in variant 1 resp. *ah*, *bh*, *ch*, *dh*, *eh*, *fh*, *gh*, *ag*, *bg*, *cg*, *dg*, *eg*, *fg*, *af*, *bf*, *cf*, *df*, *ef*, *ae*, *be*, *ce*, *de*, *ad*, *bd*, *cd*, *ac*, *bc*, *ab*, *ba*, *cb*, *ca*, *dc*, *db*, *da*, *ed*, *ec*, *eb*, *ea*, *fe*, *fd*, *fc*, *fb*, *fa*, *gf*, *ge*, *gd*, *gc*, *gb*, *ga*, *hg*, *hf*, *he*, *hd*, *hc*, *hb*, *ha* in variant 2.

Step 3:

$\mathcal{O}_{final}(\sigma)$ and $\mathcal{S}_{final}(\sigma)$ are calculated as defined in section 5.1. The final winner is alternative $a \in A$ with $ba \notin \mathcal{O}_{final}(\sigma)$ for every $b \in A \setminus \{a\}$.

5.3. Transitivity

In section 4.1, we have proven that the binary relation O, as defined in (2.2.1), is transitive. Nevertheless, it isn't intuitively clear whether also the binary relation $O_{final}(\sigma)$, as defined in section 5.1, is transitive. It seems to be possible that ties $P_\sigma[x,y] \approx_\sigma P_\sigma[y,x]$ are resolved based on different sets of *non-forbidden* links, so that the transitivity of $O_{final}(\sigma)$ doesn't follow directly from the transitivity of O.

However, in the following proof, we will see that also the binary relation $O_{final}(\sigma)$, as defined in section 5.1, is transitive. We will prove that ties $P_\sigma[x,y] \approx_\sigma P_\sigma[y,x]$ are either resolved based on the same set of *non-forbidden* links (sections 5.3.1, 5.3.4, and 5.3.5) or — in those cases, where these ties happen to be resolved based on different sets of *non-forbidden* links — they cannot violate transitivity (sections 5.3.2 and 5.3.3).

5.3.1. Part 1

Suppose, before we start declaring links *forbidden*, we have:

(5.3.1.1) $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$.

(5.3.1.2) $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$.

(5.3.1.3) $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$.

With (5.3.1.1), we get $ab \in O$ and, therefore, $ab \in O_{final}(\sigma)$.

With (5.3.1.2), we get $bc \in O$ and, therefore, $bc \in O_{final}(\sigma)$.

This situation is not possible because, when no link has been declared *forbidden*, then all paths are calculated based on the same set of *non-forbidden* links. But in section 4.1, we have proven that, when all paths are calculated based on the same set of links, then the binary relation O, as defined by $P_\sigma[x,y] \succ_\sigma P_\sigma[y,x]$, is transitive. So, with $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$, we immediately get $P_\sigma[a,c] \succ_\sigma P_\sigma[c,a]$.

5.3.2. Part 2

Suppose, before we start declaring links *forbidden*, we have:

(5.3.2.1) $P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$.

(5.3.2.2) $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$.

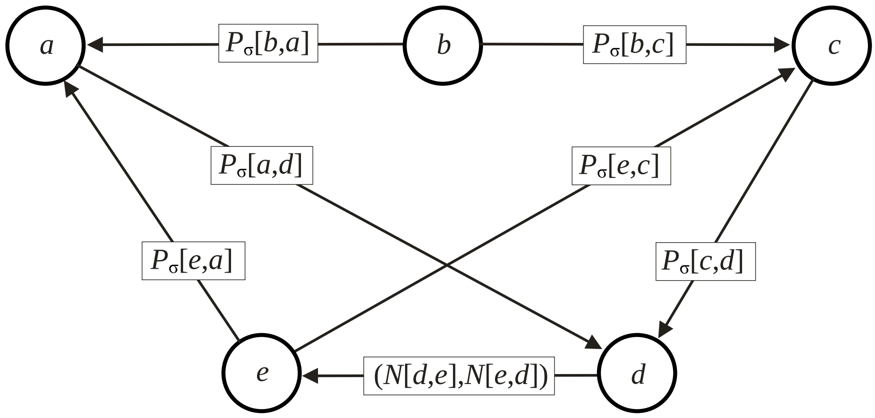
(5.3.2.3) $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$.

With (5.3.2.1), we get $ba \in O$ and, therefore, $ba \in O_{final}(\sigma)$.

With (5.3.2.2), we get $bc \in O$ and, therefore, $bc \in O_{final}(\sigma)$.

Suppose there are no pairwise links of equivalent strengths. Suppose (5.3.2.1) – (5.3.2.3). Then the weakest link in the strongest path from alternative a to alternative c and the weakest link in the strongest path from alternative c to alternative a must be the same link, say de.

Therefore, the strongest paths have the following structure:



In this case, it can actually happen that the paths are based on different sets of *non-forbidden* links. In example 8 (section 3.8), we have a situation with $P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$, $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$, and $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$ and where the link *de* is the weakest link in the strongest path from alternative *a* to alternative *c* and simultaneously the weakest link in the strongest path from alternative *c* to alternative *a*. So when we resolve $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$, the link *de* has to be declared *forbidden*. The strongest path from alternative *a* to alternative *c*, that doesn’t contain the link *de*, is *a*,(24,21),*c*. The strongest path from alternative *c* to alternative *a*, that doesn’t contain the link *de*, is *c*,(25,20),*b*,(22,23),*e*,(30,15),*a*. So $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$ is resolved to $ac \in O_{final}(\sigma)$.

Now the interesting observation is that the link *de* is also in the strongest path from alternative *b* to alternative *a*. And the strongest path *b*,(22,23),*e*, (30,15),*a* from alternative *b* to alternative *a*, that doesn’t contain the link *de*, is weaker than the strongest path *a*,(26,19),*b* from alternative *a* to alternative *b*, that doesn’t contain the link *de*. Therefore, if we had to recalculate the strengths of the strongest paths from alternative *a* to alternative *b* and from alternative *b* to alternative *a* based on the fact that the link *de* has been declared *forbidden* { what we don’t have to do, because each of (5.3.2.1) – (5.3.2.3) is resolved separately, based on its own set of *non-forbidden* links }, we would get $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$.

Furthermore, the link *de* is in the strongest path from alternative *b* to alternative *c*. And the strongest path *b*,(22,23),*e*,(32,13),*c* from alternative *b* to alternative *c*, that doesn’t contain the link *de*, is weaker than the strongest path *c*,(25,20),*b* from alternative *c* to alternative *b*, that doesn’t contain the link *de*. Therefore, if we had to recalculate the strengths of the strongest paths from alternative *b* to alternative *c* and from alternative *c* to alternative *b* based on the fact that the link *de* has been declared *forbidden*, we would get $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$.

So example 8 (section 3.8) demonstrates that it can happen that (5.3.2.1) – (5.3.2.3) are resolved based on different sets of *non-forbidden* links. However, this is not a problem because — it doesn’t matter whether $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$ is resolved to $P_\sigma[c,a] \succ_\sigma P_\sigma[a,c]$ or to $P_\sigma[c,a] \prec_\sigma P_\sigma[a,c]$ — transitivity will never be violated.

5.3.3. Part 3

Suppose, before we start declaring links *forbidden*, we have:

(5.3.3.1) $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$.

(5.3.3.2) $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$.

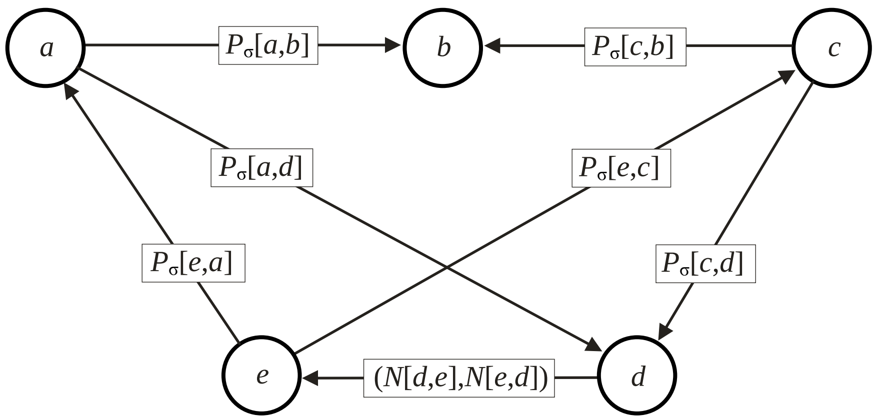
(5.3.3.3) $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$.

With (5.3.3.1), we get $ab \in \mathcal{O}$ and, therefore, $ab \in \mathcal{O}_{final}(\sigma)$.

With (5.3.3.2), we get $cb \in \mathcal{O}$ and, therefore, $cb \in \mathcal{O}_{final}(\sigma)$.

Suppose there are no pairwise links of equivalent strengths. Suppose (5.3.3.1) – (5.3.3.3). Then the weakest link in the strongest path from alternative *a* to alternative *c* and the weakest link in the strongest path from alternative *c* to alternative *a* must be the same link, say *de*.

Therefore, the strongest paths have the following structure:



In this case, it can actually happen that the paths are based on different sets of *non-forbidden* links. In example 9 (section 3.9), we have a situation with $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$, $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$, and $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$ and where the link *de* is the weakest link in the strongest path from alternative *a* to alternative *c* and simultaneously the weakest link in the strongest path from alternative *c* to alternative *a*. So when we resolve $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$, the link *de* has to be declared *forbidden*. The strongest path from alternative *a* to alternative *c*, that doesn’t contain the link *de*, is *a*,<u>(24,21)</u>,*c*. The strongest path from alternative *c* to alternative *a*, that doesn’t contain the link *de*, is *c*,(30,15),*d*,<u>(22,23)</u>,*b*,(25,20),*a*. So $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$ is resolved to $ac \in \mathcal{O}_{final}(\sigma)$.

Now the interesting observation is that the link *de* is also in the strongest path from alternative *a* to alternative *b*. And the strongest path *a*,(32,13),*d*,<u>(22,23)</u>,*b* from alternative *a* to alternative *b*, that doesn’t contain the link *de*, is weaker than the strongest path *b*,<u>(25,20)</u>,*a* from alternative *b* to alternative *a*, that doesn’t contain the link *de*. Therefore, if we had to recalculate the strengths of the strongest paths from alternative *a* to alternative *b* and from alternative *b* to alternative *a* based on the fact that the link *de* has been declared *forbidden* { what we don’t have to do, because each of (5.3.3.1) –

(5.3.3.3) is resolved separately, based on its own set of *non-forbidden* links }, we would get $P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$.

Furthermore, the link *de* is in the strongest path from alternative *c* to alternative *b*. And the strongest path *c*,(30,15),*d*,(22,23),*b* from alternative *c* to alternative *b*, that doesn’t contain the link *de*, is weaker than the strongest path *b*,(26,19),*c* from alternative *b* to alternative *c*, that doesn’t contain the link *de*. Therefore, if we had to recalculate the strengths of the strongest paths from alternative *b* to alternative *c* and from alternative *c* to alternative *b* based on the fact that the link *de* has been declared *forbidden*, we would get $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$.

So example 9 (section 3.9) demonstrates that it can happen that (5.3.3.1) – (5.3.3.3) are resolved based on different sets of *non-forbidden* links. However, this is not a problem because — it doesn’t matter whether $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$ is resolved to $P_\sigma[c,a] \succ_\sigma P_\sigma[a,c]$ or to $P_\sigma[c,a] \prec_\sigma P_\sigma[a,c]$ — transitivity will never be violated.

5.3.4. Part 4

Suppose, before we start declaring links *forbidden*, we have:

(5.3.4.1) $P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$.

(5.3.4.2) $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$.

(5.3.4.3) $P_\sigma[c,a] \succ_\sigma P_\sigma[a,c]$.

With (5.3.4.3), we get $ca \in O$ and, therefore, $ca \in O_{final}(\sigma)$.

As the tie (5.3.4.1) and the tie (5.3.4.2) are resolved separately, it seems to be possible that they are resolved based on different sets of *non-forbidden* links, so that the transitivity of $O_{final}(\sigma)$ doesn’t follow directly from the transitivity of O. It seems to be possible that the tie (5.3.4.1) is resolved to $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and that simultaneously — as other links are declared *forbidden* during the process of resolving the tie (5.3.4.2), so that the strengths of the strongest paths are determined based on different sets of *non-forbidden* links — the tie (5.3.4.2) is resolved to $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$, so that the transitivity of $O_{final}(\sigma)$ is violated. However, the following proof shows that transitivity will never be violated.

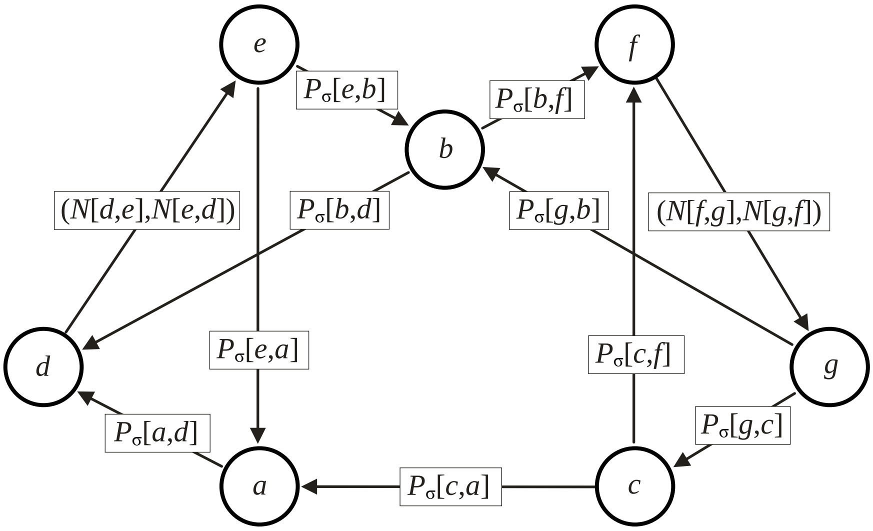
Claim:

Suppose (5.3.4.1) – (5.3.4.3) are resolved as prescribed in section 5.1. Then transitivity will never be violated.

Proof:

Suppose there are no pairwise links of equivalent strengths. Suppose (5.3.4.1) – (5.3.4.3). Then the weakest link in the strongest path from alternative *a* to alternative *b* and the weakest link in the strongest path from alternative *b* to alternative *a* must be the same link, say *de*. Furthermore, the weakest link in the strongest path from alternative *b* to alternative *c* and the weakest link in the strongest path from alternative *c* to alternative *b* must be the same link, say *fg*.

Therefore, the strongest paths have the following structure:



As *de* is the weakest link in the strongest path from alternative *a* to alternative *b*, we get

(5.3.4.4) $P_\sigma[a,d] \succ_\sigma (N[d,e],N[e,d])$.

(5.3.4.5) $P_\sigma[e,b] \succ_\sigma (N[d,e],N[e,d])$.

As *de* is the weakest link in the strongest path from alternative *b* to alternative *a*, we get

(5.3.4.6) $P_\sigma[b,d] \succ_\sigma (N[d,e],N[e,d])$.

(5.3.4.7) $P_\sigma[e,a] \succ_\sigma (N[d,e],N[e,d])$.

As *fg* is the weakest link in the strongest path from alternative *b* to alternative *c*, we get

(5.3.4.8) $P_\sigma[b,f] \succ_\sigma (N[f,g],N[g,f])$.

(5.3.4.9) $P_\sigma[g,c] \succ_\sigma (N[f,g],N[g,f])$.

As *fg* is the weakest link in the strongest path from alternative *c* to alternative *b*, we get

(5.3.4.10) $P_\sigma[c,f] \succ_\sigma (N[f,g],N[g,f])$.

(5.3.4.11) $P_\sigma[g,b] \succ_\sigma (N[f,g],N[g,f])$.

With (5.3.4.4), (5.3.4.5), (5.3.4.8), and (5.3.4.9), we get: $a \to d \to e \to b \to f \to g \to c$ is a path from alternative *a* to alternative *c* with a strength of $\min_\sigma \{ (N[d,e],N[e,d]), (N[f,g],N[g,f]) \}$. Therefore, with (5.3.4.3), we get

(5.3.4.12) $P_\sigma[c,a] \succ_\sigma \min_\sigma \{ (N[d,e],N[e,d]), (N[f,g],N[g,f]) \}$.

<u>Case 1:</u> Suppose

(5.3.4.13a) $(N[d,e],N[e,d]) \succ_{\sigma} (N[f,g],N[g,f])$.

Then, with (5.3.4.12), (5.3.4.4), (5.3.4.13a), and (5.3.4.5), we get: $c \rightarrow a \rightarrow d \rightarrow e \rightarrow b$ is a path from alternative *c* to alternative *b* with a strength of more than $(N[f,g],N[g,f])$. But this is a contradiction to the presumption that *fg* is the weakest link in the strongest path from alternative *c* to alternative *b*.

<u>Case 2:</u> Suppose

(5.3.4.13b) $(N[d,e],N[e,d]) \prec_{\sigma} (N[f,g],N[g,f])$.

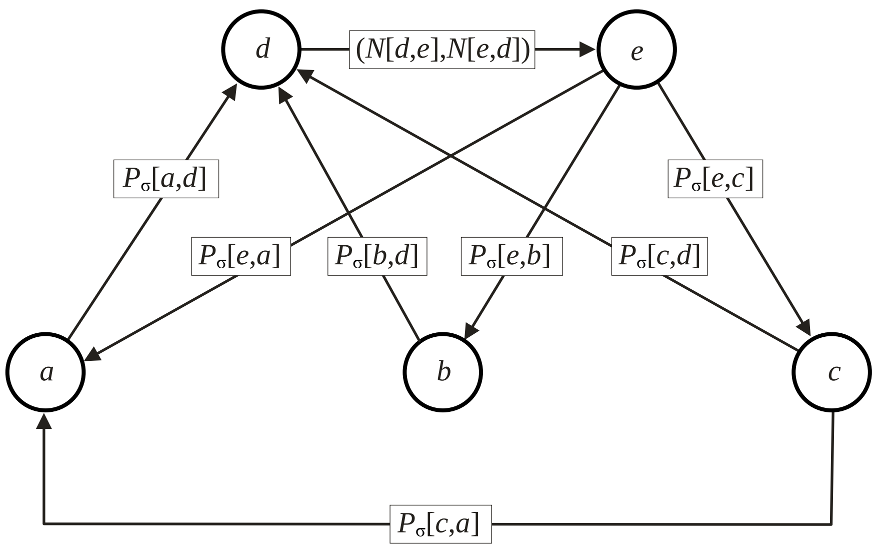
Then, with (5.3.4.8), (5.3.4.13b), (5.3.4.9), and (5.3.4.12), we get: $b \rightarrow f \rightarrow g \rightarrow c \rightarrow a$ is a path from alternative *b* to alternative *a* with a strength of more than $(N[d,e],N[e,d])$. But this is a contradiction to the presumption that *de* is the weakest link in the strongest path from alternative *b* to alternative *a*.

As (5.3.4.13a) and (5.3.4.13b) are not possible, we get

(5.3.4.13c) $(N[d,e],N[e,d]) \approx_{\sigma} (N[f,g],N[g,f])$.

As there are no links of equivalent strengths, (5.3.4.13c) means that *de* and *fg* are the same link. So to resolve (5.3.4.1) and (5.3.4.2), the same link is declared *forbidden*.

Therefore, the strongest paths have the following structure:



Without loss of generality, we can also say that the same link is declared *forbidden* in the process of resolving (5.3.4.3). The reason: With (5.3.4.12), we get that the link *de* cannot be in the strongest path from alternative *c* to alternative *a*. Therefore, the strongest path from alternative *c* to alternative *a* cannot be weakened by declaring the link *de forbidden*. The strongest path from alternative *a* to alternative *c* can be weakened by declaring the link *de forbidden*. But as we already know from (5.3.4.3) that the strongest path from alternative *c* to alternative *a* is stronger than the strongest path from alternative *a* to alternative *c*, declaring the link *de forbidden* cannot have an impact on the resolution of (5.3.4.3).

When the link *de* is declared *forbidden*, we get one of the following cases:

<u>Case A:</u> We still get $P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$. In this case, with the same argumentation as in cases 1–2 we get that the same link, say *d’e’*, is the weakest link in the strongest path from alternative *a* to alternative *b*, the weakest link in the strongest path from alternative *b* to alternative *a*, the weakest link in the strongest path from alternative *b* to alternative *c*, and the weakest link in the strongest path from alternative *c* to alternative *b*. So we can proceed with declaring the link *d’e’ forbidden* until we get one of the cases B–G.

<u>Case B:</u> We get ($P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$) or ($P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$) or ($P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$). In this case, we succeeded in resolving (5.3.4.1) – (5.3.4.3) without violating transitivity.

<u>Case C:</u> We get $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$. This case is not possible because, after the link *de* has been declared *forbidden*, (5.3.4.1) – (5.3.4.3) are still calculated based on the same set of *non-forbidden* links. So with $P_\sigma[c,a] \succ_\sigma P_\sigma[a,c]$ and $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and the transitivity, as proven in section 4.1 for cases where all paths are based on the same set of *non-forbidden* links, we would immediately get $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$.

<u>Case D:</u> We get $P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$. This case is not possible because, after the link *de* has been declared *forbidden*, (5.3.4.1) – (5.3.4.3) are still calculated based on the same set of *non-forbidden* links. So with $P_\sigma[c,a] \succ_\sigma P_\sigma[a,c]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$ and the transitivity, as proven in section 4.1 for cases where all paths are based on the same set of *non-forbidden* links, we would immediately get $P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$.

Case E: We get $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$. This case is not possible because, after the link *de* has been declared *forbidden*, (5.3.4.1) – (5.3.4.3) are still calculated based on the same set of *non-forbidden* links. So $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$, $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$, and $P_\sigma[c,a] \succ_\sigma P_\sigma[a,c]$ together violate transitivity, as proven in section 4.1 for cases where all paths are based on the same set of *non-forbidden* links.

Case F: We get $P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$. This case is identical to the situation in section 5.3.2. It is possible that $P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ is resolved based on a different set of *non-forbidden* links. However, this is not a problem because — it doesn’t matter whether $P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ is resolved to $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ or to $P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ — transitivity will never be violated.

Case G: We get $P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$. This case is identical to the situation in section 5.3.3. It is possible that $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$ is resolved based on a different set of *non-forbidden* links. However, this is not a problem because — it doesn’t matter whether $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$ is resolved to $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$ or to $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$ — transitivity will never be violated.

The following table shows that cases A–G cover all possible combinations. Therefore, it has been proven for every possible situation that, when we resolve (5.3.4.1) – (5.3.4.3) as prescribed in section 5.1, then transitivity will never be violated.

$P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$	→ case A
$P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$	→ case D
$P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$	→ case F
$P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$	→ case C
$P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$	→ case E
$P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$	→ case B
$P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$	→ case G
$P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$	→ case B
$P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$	→ case B

□

5.3.5. Part 5

Suppose, before we start declaring links *forbidden*, we have:

(5.3.5.1) $P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$.

(5.3.5.2) $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$.

(5.3.5.3) $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$.

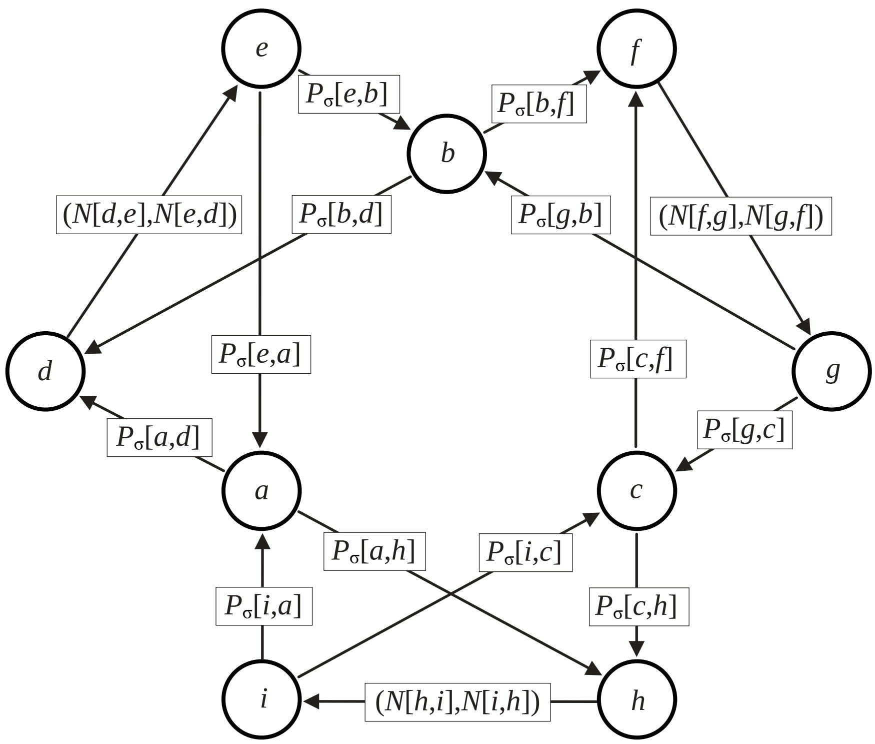
Claim:

Suppose (5.3.5.1) – (5.3.5.3) are resolved as prescribed in section 5.1. Then transitivity will never be violated.

Proof:

Suppose there are no pairwise links of equivalent strengths. Suppose (5.3.5.1) – (5.3.5.3). Then the weakest link in the strongest path from alternative *a* to alternative *b* and the weakest link in the strongest path from alternative *b* to alternative *a* must be the same link, say *de*. Furthermore, the weakest link in the strongest path from alternative *b* to alternative *c* and the weakest link in the strongest path from alternative *c* to alternative *b* must be the same link, say *fg*. Furthermore, the weakest link in the strongest path from alternative *c* to alternative *a* and the weakest link in the strongest path from alternative *a* to alternative *c* must be the same link, say *hi*.

Therefore, the strongest paths have the following structure:



As *de* is the weakest link in the strongest path from alternative *a* to alternative *b*, we get

(5.3.5.4) $P_\sigma[a,d] \succ_\sigma (N[d,e],N[e,d])$.

(5.3.5.5) $P_\sigma[e,b] \succ_\sigma (N[d,e],N[e,d])$.

As *de* is the weakest link in the strongest path from alternative *b* to alternative *a*, we get

(5.3.5.6) $P_\sigma[b,d] \succ_\sigma (N[d,e],N[e,d])$.

(5.3.5.7) $P_\sigma[e,a] \succ_\sigma (N[d,e],N[e,d])$.

As *fg* is the weakest link in the strongest path from alternative *b* to alternative *c*, we get

(5.3.5.8) $P_\sigma[b,f] \succ_\sigma (N[f,g],N[g,f])$.

(5.3.5.9) $P_\sigma[g,c] \succ_\sigma (N[f,g],N[g,f])$.

As *fg* is the weakest link in the strongest path from alternative *c* to alternative *b*, we get

(5.3.5.10) $P_\sigma[c,f] \succ_\sigma (N[f,g],N[g,f])$.

(5.3.5.11) $P_\sigma[g,b] \succ_\sigma (N[f,g],N[g,f])$.

As *hi* is the weakest link in the strongest path from alternative *c* to alternative *a*, we get

(5.3.5.12) $P_\sigma[c,h] \succ_\sigma (N[h,i],N[i,h])$.

(5.3.5.13) $P_\sigma[i,a] \succ_\sigma (N[h,i],N[i,h])$.

As *hi* is the weakest link in the strongest path from alternative *a* to alternative *c*, we get

(5.3.5.14) $P_\sigma[a,h] \succ_\sigma (N[h,i],N[i,h])$.

(5.3.5.15) $P_\sigma[i,c] \succ_\sigma (N[h,i],N[i,h])$.

Case 1: Suppose

(5.3.5.16a) $(N[d,e],N[e,d]) \prec_\sigma (N[f,g],N[g,f])$.

(5.3.5.17a) $(N[d,e],N[e,d]) \prec_\sigma (N[h,i],N[i,h])$.

Then, with (5.3.5.14), (5.3.5.17a), (5.3.5.15), (5.3.5.10), (5.3.5.16a), and (5.3.5.11), we get: $a \to h \to i \to c \to f \to g \to b$ is a path from alternative a to alternative b with a strength of more than $(N[d,e],N[e,d])$. But this is a contradiction to the presumption that de is the weakest link in the strongest path from alternative a to alternative b.

Similarly, with (5.3.5.8), (5.3.5.16a), (5.3.5.9), (5.3.5.12), (5.3.5.17a), and (5.3.5.13), we get: $b \to f \to g \to c \to h \to i \to a$ is a path from alternative b to alternative a with a strength of more than $(N[d,e],N[e,d])$. But this is a contradiction to the presumption that de is the weakest link in the strongest path from alternative b to alternative a.

Case 2: Suppose

(5.3.5.16b) $(N[f,g],N[g,f]) \prec_\sigma (N[d,e],N[e,d])$.

(5.3.5.17b) $(N[f,g],N[g,f]) \prec_\sigma (N[h,i],N[i,h])$.

Then, with (5.3.5.6), (5.3.5.16b), (5.3.5.7), (5.3.5.14), (5.3.5.17b), and (5.3.5.15), we get: $b \to d \to e \to a \to h \to i \to c$ is a path from alternative b to alternative c with a strength of more than $(N[f,g],N[g,f])$. But this is a contradiction to the presumption that fg is the weakest link in the strongest path from alternative b to alternative c.

Similarly, with (5.3.5.12), (5.3.5.17b), (5.3.5.13), (5.3.5.4), (5.3.5.16b), and (5.3.5.5), we get: $c \to h \to i \to a \to d \to e \to b$ is a path from alternative c to alternative b with a strength of more than $(N[f,g],N[g,f])$. But this is a contradiction to the presumption that fg is the weakest link in the strongest path from alternative c to alternative b.

Case 3: Suppose

(5.3.5.16c) $(N[h,i],N[i,h]) \prec_\sigma (N[d,e],N[e,d])$.

(5.3.5.17c) $(N[h,i],N[i,h]) \prec_\sigma (N[f,g],N[g,f])$.

Then, with (5.3.5.10), (5.3.5.17c), (5.3.5.11), (5.3.5.6), (5.3.5.16c), and (5.3.5.7), we get: $c \to f \to g \to b \to d \to e \to a$ is a path from alternative c to alternative a with a strength of more than $(N[h,i],N[i,h])$. But this is a contradiction to the presumption that hi is the weakest link in the strongest path from alternative c to alternative a.

Similarly, with (5.3.5.4), (5.3.5.16c), (5.3.5.5), (5.3.5.8), (5.3.5.17c), and (5.3.5.9), we get: $a \to d \to e \to b \to f \to g \to c$ is a path from alternative a to alternative c with a strength of more than $(N[h,i],N[i,h])$. But this is a contradiction to the presumption that hi is the weakest link in the strongest path from alternative a to alternative c.

With cases 1–3, we get that none of the links *de*, *fg*, *hi* can be weaker than the other two links. Without loss of generality, we can presume that the link *hi* is the strongest one of the links *de, fg*, *hi*. So we get

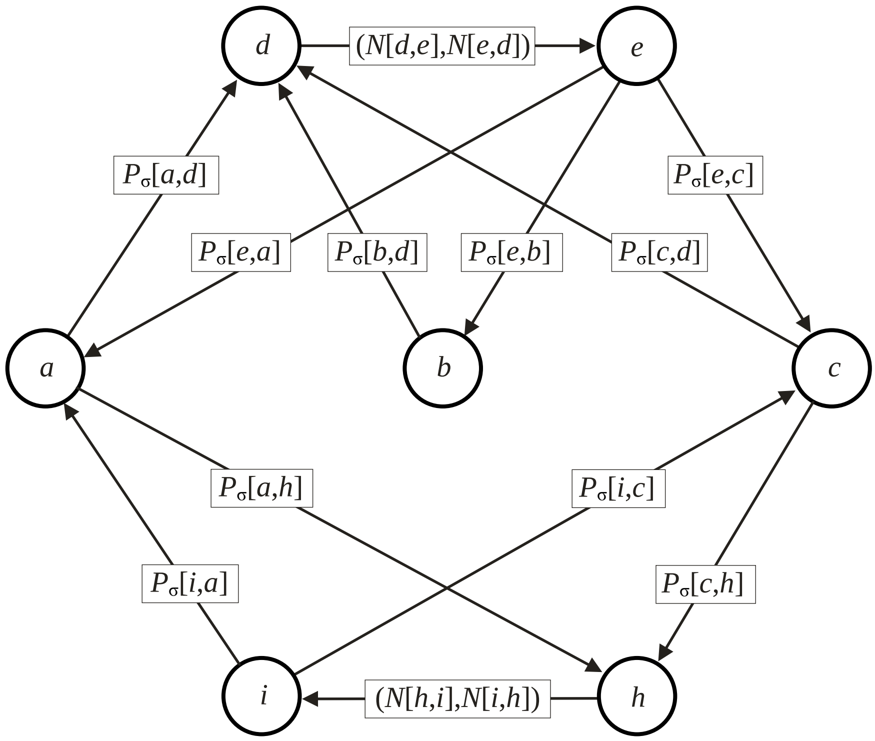
(5.3.5.18) $(N[d,e],N[e,d]) \approx_\sigma (N[f,g],N[g,f]) \precsim_\sigma (N[h,i],N[i,h])$.

We can ignore the case $(N[d,e],N[e,d]) \approx_\sigma (N[f,g],N[g,f]) \approx_\sigma (N[h,i], N[i,h])$ because in this case the links *de*, *fg*, *hi* are the same link so that for each of (5.3.5.1) – (5.3.5.3) the same link is declared *forbidden* first so that, afterwards, each of (5.3.5.1) – (5.3.5.3) is still resolved based on the same set of *non-forbidden* links.

So without loss of generality, we get

(5.3.5.19) $(N[d,e],N[e,d]) \approx_\sigma (N[f,g],N[g,f]) \prec_\sigma (N[h,i],N[i,h])$.

As there are no links of equivalent strengths, (5.3.5.19) means that the link *de* and the link *fg* must be the same link. Therefore, the strongest paths have the following structure:



Without loss of generality, we can also say that, when we resolve (5.3.5.1) – (5.3.5.3), then, at each stage, the weakest of the weakest links of the current strongest paths is declared *forbidden*. So in our situation, the link *de* is declared *forbidden* next.

Since $(N[d,e],N[e,d]) \prec_\sigma (N[h,i],N[i,h]) \approx_\sigma P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$, the link *de* cannot be in the strongest path from alternative *c* to alternative *a* or in the strongest path from alternative *a* to alternative *c*. Therefore, declaring the link *de forbidden* cannot have an impact on the strongest path from alternative *c* to alternative *a* or on the strongest path from alternative *a* to alternative *c*.

When the link *de* is declared *forbidden*, we get one of the following cases:

Case A: We still get $P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$. In this case, with the same argumentation as in cases 1–2 we get that the same link, say *d’e’*, is the weakest link in the strongest path from alternative *a* to alternative *b*, the weakest link in the strongest path from alternative *b* to alternative *a*, the weakest link in the strongest path from alternative *b* to alternative *c*, and the weakest link in the strongest path from alternative *c* to alternative *b*. So we can proceed with declaring the link *d’e’ forbidden* until we get one of the cases B–F.

Case B: We get $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$. This case is not possible because, after the link *de* has been declared *forbidden*, (5.3.5.1) – (5.3.5.3) are still calculated based on the same set of *non-forbidden* links. With $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$ and the transitivity, as proven in section 4.1 for cases where all paths are based on the same set of *non-forbidden* links, we would immediately get $P_\sigma[c,a] \prec_\sigma P_\sigma[a,c]$. But this is a contradiction to the fact that the link *de* cannot have been in the strongest path from alternative *c* to alternative *a* or in the strongest path from alternative *a* to alternative *c*, so that declaring the link *de forbidden* cannot have an impact on $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$.

Case C: We get $P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$. This case is not possible because, after the link *de* has been declared *forbidden*, (5.3.5.1) – (5.3.5.3) are still calculated based on the same set of *non-forbidden* links. With $P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$ and the transitivity, as proven in section 4.1 for cases where all paths are based on the same set of *non-forbidden* links, we would immediately get $P_\sigma[c,a] \succ_\sigma P_\sigma[a,c]$. But this is a contradiction to the fact that the link *de* cannot have been in the strongest path from alternative *c* to alternative *a* or in the strongest path from alternative *a* to alternative *c*, so that declaring the link *de forbidden* cannot have an impact on $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$.

Case D: We get ($P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$) or ($P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$) or ($P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$) or ($P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$). This case is not possible because we have seen in (5.3.4.13a) – (5.3.4.13c) that, when we have a situation with $P_\sigma[x,y] \approx_\sigma P_\sigma[y,x]$, $P_\sigma[y,z] \approx_\sigma P_\sigma[z,y]$, and $P_\sigma[z,x] \succ_\sigma P_\sigma[x,z]$, then the weakest link in the strongest path from alternative *x* to alternative *y*, the weakest link in the strongest path from alternative *y* to alternative *x*, the weakest link in the strongest path from alternative *y* to alternative *z*, and the weakest link in the strongest path from alternative *z* to alternative *y* must be the same link. But this is not possible because (5.3.5.19) says that the link *hi* is stronger than the link *de*.

Case E: We get $P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$. This case is identical to the situation in section 5.3.2. It is possible that $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$ is resolved based on a different set of *non-forbidden* links. However, this is not a problem because –– it doesn't matter whether $P_\sigma[a,c] \approx_\sigma P_\sigma[c,a]$ is resolved to $P_\sigma[a,c] \succ_\sigma P_\sigma[c,a]$ or to $P_\sigma[a,c] \prec_\sigma P_\sigma[c,a]$ –– transitivity will never be violated.

Case F: We get $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$. This case is identical to the situation in section 5.3.3. It is possible that $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$ is resolved based on a different set of *non-forbidden* links. However, this is not a problem because –– it doesn't matter whether $P_\sigma[a,c] \approx_\sigma P_\sigma[c,a]$ is resolved to $P_\sigma[a,c] \succ_\sigma P_\sigma[c,a]$ or to $P_\sigma[a,c] \prec_\sigma P_\sigma[c,a]$ –– transitivity will never be violated.

The following table shows that cases A–F cover all possible combinations. Therefore, it has been proven for every possible situation that, when we resolve (5.3.5.1) – (5.3.5.3) as prescribed in section 5.1, then transitivity will never be violated.

$P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$	→ case A
$P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$	→ case D
$P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$	→ case D
$P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$	→ case D
$P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$	→ case B
$P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$	→ case F
$P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$	→ case D
$P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$	→ case E
$P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$	→ case C

□

5.4. Analysis

An election method is simply a mapping from some input to some output. In section 2.1, we presumed that the output is (1) a strict partial order $\mathcal{O}$ on A and (2) a set $\emptyset \neq \mathcal{S} \subseteq A$ of potential winners. In the probabilistic framework, the output of an election method is a probability distribution $p[\mathcal{O}] \in \mathbb{R}$ on $\mathcal{LO}_A$, where $\mathcal{LO}_A$ is the set of linear orders on A.

We get

(5.4.1) $\forall\ \mathcal{O} \in \mathcal{LO}_A$: $p[\mathcal{O}] \geq 0$.

(5.4.2) $\sum\ (\ p[\mathcal{O}]\ |\ \mathcal{O} \in \mathcal{LO}_A\ ) = 1$.

Suppose $q[a,b] \in \mathbb{R}$ is the probability for $ab \in \mathcal{O}$ (i.e. the probability that alternative $a \in A$ is ranked ahead of alternative $b \in A \setminus \{a\}$ in the collective ranking $\mathcal{O}$).

Then, we get

(5.4.3) $q[a,b] := \sum\ (\ p[\mathcal{O}]\ |\ \mathcal{O} \in \mathcal{LO}_A$ with $ab \in \mathcal{O}\ )$.

(5.4.4) $\forall\ a,b \in A$: $q[a,b] \geq 0$.

(5.4.5) $\forall\ a,b \in A$: $q[a,b] + q[b,a] = 1$.

Suppose $r[a] \in \mathbb{R}$ is the probability that alternative $a \in A$ is elected.

Then, we get

(5.4.6) $r[a] := \sum\ (\ p[\mathcal{O}]\ |\ \mathcal{O} \in \mathcal{LO}_A$ with $ab \in \mathcal{O}$ for all $b \in A \setminus \{a\}\ )$.

(5.4.7) $\forall\ a \in A$: $r[a] \geq 0$.

(5.4.8) $\sum\ (\ r[a]\ |\ a \in A\ ) = 1$.

Resolvability

Definition:

An election method satisfies the *resolvability criterion* if (for every given number of alternatives) the proportion of profiles without a unique linear order (i.e. without a linear order $\mathcal{O} \in \mathcal{LO}_A$ with $p[\mathcal{O}] = 1$) tends to zero as the number of voters in the profile tends to infinity.

Claim:

If $\succ_D$ satisfies (2.1.1), then the Schulze method $\mathcal{O}_{final}(\sigma)$, as defined in sections 5.1, with the TBRL $\succ_\sigma$, as defined in section 5.2, satisfies the resolvability criterion.

Proof (overview):

1. Suppose the number of alternatives is fixed. We prove that, when the number of voters in the profile tends to infinity, the probability, that there are links with equivalent strengths, goes to zero. So the probability, that there are links *ef* and *gh* with $ef \approx_\sigma gh$, goes to zero.

2. We prove that (1) the link *ij* cannot be in the strongest path from alternative *j* to alternative *i* and (2) the link *ji* cannot be in the strongest path from alternative *i* to alternative *j*. Therefore, when we resolve the tie $P_\sigma[i,j] \approx_\sigma P_\sigma[j,i]$, it can neither happen that the link *ij* is declared *forbidden* nor that the link *ji* is declared *forbidden*. Therefore, in worst case, when there are no other paths of *non-forbidden* links anymore, $P_\sigma[i,j] \approx_\sigma P_\sigma[j,i]$ is resolved to $ij \in \mathcal{O}$ when $ij \succ_\sigma ji$ and to $ji \in \mathcal{O}$ when $ij \prec_\sigma ji$. So the algorithm in section 5.1 always terminates before all links have been declared *forbidden*.

Remark:

When there is a unique linear order (i.e. a linear order $\mathcal{O} \in \mathcal{LO}_A$ with $p[\mathcal{O}] = 1$) then, with (5.4.6), we get that there is also a unique winner (i.e. an alternative $a \in A$ with $r[a] = 1$):

$$(\exists\, \mathcal{O} \in \mathcal{LO}_A\colon p[\mathcal{O}] = 1) \Rightarrow (\exists\, a \in A\colon r[a] = 1).$$

Pareto

In the probabilistic framework, the *Pareto criterion* says that, when no voter strictly prefers alternative $b \in A$ to alternative $a \in A$ [see (5.4.9)] and at least one voter strictly prefers alternative a to alternative b [see (5.4.10)], then $r[b] = 0$.

Definition:

An election method satisfies the *Pareto criterion* if the following holds:

Suppose:

(5.4.9) $\forall\ v \in V\colon a \gtrsim_v b.$

(5.4.10) $\exists\ v \in V\colon a \succ_v b.$

Then:

(5.4.11) $q[a,b] = 1.$

(5.4.12) $r[b] = 0.$

Claim:

If $\succ_D$ satisfies (2.1.1), then the Schulze method $\mathcal{O}_{final}(\sigma)$, as defined in sections 5.1, with the TBRL $\succ_\sigma$, as defined in section 5.2, satisfies the Pareto criterion.

Proof (overview):

We prove

(5.4.13) $a \succ_\mu b$ with certainty.

With (4.3.2.8), (5.2.1), (5.2.6a), and (5.2.6b), we prove

(5.4.14) $\forall\ e \in A \setminus \{a,b\}\colon ae \succ_\sigma be$ with certainty.

With (4.3.2.9), (5.2.1), (5.2.5a), and (5.2.5b), we prove

(5.4.15) $\forall\ e \in A \setminus \{a,b\}\colon eb \succ_\sigma ea$ with certainty.

With (2.1.1), (5.2.1), (5.4.9), and (5.4.10), we prove

(5.4.16) $ab \succ_\sigma ba$ with certainty.

With (5.4.14), (5.4.15), and (5.4.16), we prove

(5.4.17) $ab \in \mathcal{O}$ with certainty.

Reversal Symmetry

In the probabilistic framework, *reversal symmetry* says that, when $\succ_v$ is reversed for all $v \in V$, then $r^{\text{old}}[a] + r^{\text{new}}[a] \leq 1$ for all $a \in A$. Otherwise, if $r^{\text{old}}[a] + r^{\text{new}}[a]$ was larger than 1 for some alternative $a \in A$, then this would mean that, with a probability of at least $r^{\text{old}}[a] + r^{\text{new}}[a] - 1 > 0$, alternative a is identified as best alternative and, simultaneously, identified as worst alternative.

Suppose $\mathcal{O}^{reverse} \in \mathcal{LO}_A$ is the reversal of $\mathcal{O} \in \mathcal{LO}_A$.

That means:

(5.4.18) $\forall\, a,b \in A$: $ab \in \mathcal{O} \Leftrightarrow ba \in \mathcal{O}^{reverse}$.

Definition:

An election method satisfies *reversal symmetry* if the following holds:

Suppose:

(5.4.19) $\forall\, e,f \in A\ \forall\, v \in V$: $e \succ_v^{\text{old}} f \Leftrightarrow f \succ_v^{\text{new}} e$.

Then:

(5.4.20) $\forall\, \mathcal{O} \in \mathcal{LO}_A$: $p^{\text{old}}[\mathcal{O}] = p^{\text{new}}[\mathcal{O}^{reverse}]$.

(5.4.21) $\forall\, a,b \in A$: $q^{\text{old}}[a,b] = q^{\text{new}}[b,a]$.

(5.4.22) $\forall\, a \in A$: $r^{\text{old}}[a] + r^{\text{new}}[a] \leq 1$.

Claim:

Suppose $\succ_D$ satisfies (2.1.2). Suppose, for every $(i,j),(m,n) \in A \times A$, there is at least one voter $v \in V$ with (5.2.1). Then the Schulze method $\mathcal{O}_{final}(\sigma)$, as defined in sections 5.1, with the TBRL $\succ_\sigma$, as defined in section 5.2, satisfies reversal symmetry.

Proof (overview):

Suppose, for every $(i,j),(m,n) \in A \times A$, there is at least one voter $v \in V$ with (5.2.1). Then the TBRL $\succ_\sigma$, as determined in step 1 of section 5.2, is already linear.

Furthermore, (2.1.2) guarantees that, when $\succ_v$ is reversed for all $v \in V$, also the TBRL $\succ_\sigma$, as determined in step 1 of section 5.2, is reversed.

So the probability that $\mathcal{O}$ is chosen in the original situation is identical to the probability that $\mathcal{O}^{reverse}$ is chosen in the reversed situation. As we have presumed in section 2.1 that there are at least 2 alternatives in A, $a \in A$ cannot be the maximum element of $\mathcal{O}$ and simultaneously the maximum element of $\mathcal{O}^{reverse}$. Therefore, we get (5.4.22).

Monotonicity

In the probabilistic framework, *monotonicity* says that, when some voters rank alternative $a \in A$ higher [see (4.5.1) and (4.5.2)] without changing the order in which they rank the other alternatives relatively to each other [see (4.5.3)], then $r[a]$ must not decrease.

Definition:

An election method satisfies *monotonicity* if the following holds:

Suppose $a \in A$. Suppose the ballots are modified as described in (4.5.1) – (4.5.3). Then

(5.4.23) $\forall \emptyset \neq B \subseteq A \setminus \{a\}$:

$\sum (p^{\text{old}}[O] \mid O \in \mathcal{L}O_A \text{ with } ab \in O \text{ for all } b \in B)$

$\leq \sum (p^{\text{new}}[O] \mid O \in \mathcal{L}O_A \text{ with } ab \in O \text{ for all } b \in B)$.

(5.4.24) $\forall\, b \in A \setminus \{a\}$: $q^{\text{old}}[a,b] \leq q^{\text{new}}[a,b]$.

(5.4.25) $r^{\text{old}}[a] \leq r^{\text{new}}[a]$.

Claim:

If $\succ_D$ satisfies (2.1.1), then the Schulze method $O_{final}(\sigma)$, as defined in sections 5.1, with the TBRL $\succ_\sigma$, as defined in section 5.2, satisfies monotonicity.

Proof (overview):

We prove, that when the ballots are modified as described in (4.5.1) – (4.5.3), then links af with $f \in A \setminus \{a\}$ can only rise in the TBRL $\succ_\sigma$ compared to other links eg with $e \in A \setminus \{a\}$ and $g \in A \setminus \{e\}$. Links fa with $f \in A \setminus \{a\}$ can only fall in the TBRL $\succ_\sigma$ compared to other links eg with $g \in A \setminus \{a\}$ and $e \in A \setminus \{g\}$. Links eg with $e \in A \setminus \{a\}$ and $g \in A \setminus \{a,e\}$ neither rise nor fall in the TBRL $\succ_\sigma$ compared to other links ij with $i \in A \setminus \{a\}$ and $j \in A \setminus \{a,i\}$.

The rest of the proof is identical to the proof in section 4.5.

Independence of Clones

Definition:

An election method is *independent of clones* if the following holds:

Suppose $d \in A^{old}$. Suppose $A^{new} := (A^{old} \cup K) \setminus \{d\}$.

Suppose alternative d is replaced by the set of alternatives K in such a manner that (4.6.1) – (4.6.3) are satisfied.

Then:

(5.4.26) $\forall\, \mathcal{O}_1 \in \mathcal{LO}_{(A^{old} \setminus \{d\})}\ \forall\, B \subseteq A^{old} \setminus \{d\}\ \forall\, g \in K$:

$p^{old}[\mathcal{O}]$ for $\mathcal{O} \in \mathcal{LO}_{A^{old}}$ with
(1) $\mathcal{O}_1 \subset \mathcal{O}$ and
(2) $ad \in \mathcal{O}$ for all $a \in B$ and
(3) $db \in \mathcal{O}$ for all $b \notin B$

$= \sum (p^{new}[\mathcal{O}] \mid \mathcal{O} \in \mathcal{LO}_{A^{new}}$ with
(1) $\mathcal{O}_1 \subset \mathcal{O}$ and
(2) $ag \in \mathcal{O}$ for all $a \in B$ and
(3) $gb \in \mathcal{O}$ for all $b \notin B$).

(5.4.27) $\forall\, a,b \in A^{old} \setminus \{d\}$: $q^{old}[a,b] = q^{new}[a,b]$.

(5.4.28) $\forall\, a \in A^{old} \setminus \{d\}\ \forall\, g \in K$: $q^{old}[a,d] = q^{new}[a,g]$.

(5.4.29) $\forall\, a \in A^{old} \setminus \{d\}$:
$(((r^{old}[a] = 0) \vee (\exists\, v \in V: a \not\approx_v^{old} d)) \Rightarrow (r^{old}[a] = r^{new}[a]))$.

Remark:

The presumption $((r^{old}[a] = 0) \vee (\exists\, v \in V: a \not\approx_v^{old} d))$ is needed to exclude situations where alternative a is chosen with positive probability (i.e.: $r^{old}[a] > 0$) and every voter is indifferent between alternative a and alternative d (i.e.: $a \approx_v^{old} d$ for every $v \in V$). In those situations, alternative a and alternative d are necessarily chosen with the same probability (i.e.: $r^{old}[a] = r^{old}[d]$). When alternative d is replaced by a set K of more than one alternative in such a manner that (4.6.1) – (4.6.3) are satisfied then, again, every alternative in $(K \cup \{a\})$ is necessarily chosen with the same probability (i.e.: $r^{new}[a] = r^{new}[g]$ for every $g \in K$), so that the probability, that alternative a is chosen, necessarily drops (i.e.: $r^{old}[a] > r^{new}[a]$).

Claim:

The Schulze method $\mathcal{O}_{final}(\sigma)$, as defined in sections 5.1, with the TBRL $\succ_\sigma$, as defined in section 5.2, is independent of clones.

Proof (overview):

We prove that all the alternatives $g \in K$ are ranked in a consecutive manner in the TBRC $\succ_{\mu}$. We then prove that, for every $a \in A^{\text{old}} \setminus \{d\}$, all the links ag with $g \in K$ are ranked in a consecutive manner in the TBRL $\succ_{\sigma}$. We further prove that, for every $a \in A^{\text{old}} \setminus \{d\}$, all the links ga with $g \in K$ are ranked in a consecutive manner in the TBRL $\succ_{\sigma}$.

The rest of the proof is identical to the proof in section 4.6.

Smith

Definition:

An election method satisfies *Smith* if the following holds:

Suppose (4.7.1) and (4.7.2).

Then we get:

(5.4.30) $\forall\, a \in B_1\ \forall\, b \in B_2{:}\ q[a,b] = 1.$

(5.4.31) $\sum (\, r[a] \mid a \in B_1 \,) = 1.$

An election method satisfies *Smith-IIA* if the following holds:

Suppose (4.7.1) and (4.7.2).

Suppose $d \in B_2$ is removed. Then we get:

(5.4.32) $\forall\, \mathcal{O}_1 \in \mathcal{LO}_{B_1}{:}$
$\sum (\, p^{\text{old}}[\mathcal{O}] \mid \mathcal{O} \in \mathcal{LO}_A \text{ with } \mathcal{O}_1 \subset \mathcal{O} \,) =$
$\sum (\, p^{\text{new}}[\mathcal{O}] \mid \mathcal{O} \in \mathcal{LO}_{(A \setminus \{d\})} \text{ with } \mathcal{O}_1 \subset \mathcal{O} \,).$

(5.4.33) $\forall\, a,b \in B_1{:}\ q^{\text{old}}[a,b] = q^{\text{new}}[a,b].$

(5.4.34) $\forall\, a \in B_1{:}\ r^{\text{old}}[a] = r^{\text{new}}[a].$

Suppose $d \in B_1$ is removed. Then we get:

(5.4.35) $\forall\, \mathcal{O}_1 \in \mathcal{LO}_{B_2}{:}$
$\sum (\, p^{\text{old}}[\mathcal{O}] \mid \mathcal{O} \in \mathcal{LO}_A \text{ with } \mathcal{O}_1 \subset \mathcal{O} \,) =$
$\sum (\, p^{\text{new}}[\mathcal{O}] \mid \mathcal{O} \in \mathcal{LO}_{(A \setminus \{d\})} \text{ with } \mathcal{O}_1 \subset \mathcal{O} \,).$

(5.4.36) $\forall\, a,b \in B_2{:}\ q^{\text{old}}[a,b] = q^{\text{new}}[a,b].$

Claim:

If $\succ_D$ satisfies (2.1.5), then the Schulze method $\mathcal{O}_{final}(\sigma)$, as defined in sections 5.1, with the TBRL $\succ_{\sigma}$, as defined in section 5.2, satisfies Smith and Smith-IIA.

Proof (overview):

The proof is identical to the proofs in section 4.7.

Runtime

The runtime to calculate the pairwise matrix is $O(N \cdot (C^2))$.

The runtime to calculate the TBRL is $O(N \cdot (C^4))$ because, in worst case, $O(N)$ ballots have to be picked and, each time, $O(C^2)$ links are compared with $O(C^2)$ other links, according to (5.2.1).

The runtime to calculate a complete ranking, as defined in section 5.1, is $O(C^7)$ because, in worst case, there are $O(C^2)$ pairwise ties "$P_\sigma[m,n] \approx_\sigma P_\sigma[n,m]$" (line 54). In worst case, $O(C^2)$ links have to be declared *forbidden* to resolve a pairwise tie. Each time, the runtime of the Floyd algorithm to calculate the strength of the strongest path from every alternative to every other alternative is $O(C^3)$.

On closer examination, to resolve the pairwise tie "$P_\sigma[m,n] \approx_\sigma P_\sigma[n,m]$", it is not necessary to calculate the strength of the strongest path from every alternative to every other alternative. It is sufficient to calculate the strength of the strongest path from alternative *m* to alternative *n* and the strength of the strongest path from alternative *n* to alternative *m*. This can be done with the Dijkstra algorithm in a runtime $O(C^2)$.

Therefore, the runtime to calculate a complete ranking, as defined in section 5.1, reduces to $O(C^6)$.

Thus, the total runtime to calculate the binary relation $\mathcal{O}$, as defined in section 5, is $O(N \cdot (C^4) + C^6)$.

6. Definition of the Strength of a Pairwise Link

6.1. Winning Votes

There has been some debate about how to define $\succ_D$ when it is presumed that on the one side each voter has a sincere linear order of the alternatives, but on the other side some voters cast only a strict weak order because of strategic considerations. We got to the conclusion that the strength ($N[e,f]$, $N[f,e]$) of the pairwise link $ef \in A \times A$ should be measured by *winning votes*, i.e. primarily by the support $N[e,f]$ of this link and secondarily by the opposition $N[f,e]$ to this link.

$(N[e,f],N[f,e]) \succ_{win} (N[g,h],N[h,g])$ if and only if at least one of the following conditions is satisfied:

1. $N[e,f] > N[f,e]$ and $N[g,h] \leq N[h,g]$.
2. $N[e,f] \geq N[f,e]$ and $N[g,h] < N[h,g]$.
3. $N[e,f] > N[f,e]$ and $N[g,h] > N[h,g]$ and $N[e,f] > N[g,h]$.
4. $N[e,f] > N[f,e]$ and $N[g,h] > N[h,g]$ and $N[e,f] = N[g,h]$ and $N[f,e] < N[h,g]$.
5. $N[e,f] < N[f,e]$ and $N[g,h] < N[h,g]$ and $N[f,e] < N[h,g]$.
6. $N[e,f] < N[f,e]$ and $N[g,h] < N[h,g]$ and $N[f,e] = N[h,g]$ and $N[e,f] > N[g,h]$.

Suppose $a,b \in A$. Suppose $R_1[a] := \| \{ v \in V \mid \forall c \in A \setminus \{a\}: a \succ_v c \} \|$ is the number of voters who strictly prefer alternative a to every other alternative. Suppose $R_2[b] := \| \{ v \in V \mid \exists c \in A \setminus \{b\}: b \succ_v c \} \|$ is the number of voters who strictly prefer alternative b to at least one other alternative. Suppose $R_1[a] > R_2[b]$. Then *Woodall’s plurality criterion* says: $b \notin \mathcal{S}$. Woodall (1997) writes: “If some candidate b has strictly fewer votes in total than some other candidate a has first-preference votes, then candidate b should not be elected.”

Claim:

If $\succ_{win}$ is being used, then the Schulze method satisfies Woodall’s plurality criterion.

Proof:

Suppose

(6.1.1) $R_1[a] > R_2[b]$.

With (6.1.1) and the definition for $\succ_{win}$, we get

(6.1.2) $(R_1[a],R_2[b]) \succ_{win} (R_2[b],0)$.

With the definitions for $R_1[a]$ and $R_2[b]$, we get

(6.1.3) $N[a,b] \geq R_1[a]$.

(6.1.4) $N[b,a] \leq R_2[b]$.

With (6.1.3), (6.1.4), and the definition for $\succ_{win}$, we get

(6.1.5) $(N[a,b],N[b,a]) \succsim_{win} (R_1[a],R_2[b])$.

With the definition for $R_2[b]$, we get

(6.1.6) $\forall c \in A \setminus \{b\}: N[b,c] \leq R_2[b]$.

With (6.1.6) and the definition for $\succ_{win}$, we get

(6.1.7) $\forall c \in A \setminus \{b\}: (N[b,c],N[c,b]) \precsim_{win} (R_2[b],0)$.

With (2.2.6) and (6.1.7), we get

(6.1.8) $P_{win}[b,a] \precsim_{win} (R_2[b],0)$.

With (2.2.3), (6.1.5), (6.1.2), and (6.1.8), we get

(6.1.9) $P_{win}[a,b] \succsim_{win} (N[a,b],N[b,a]) \succsim_{win} (R_1[a],R_2[b]) \succ_{win} (R_2[b],0) \succsim_{win} P_{win}[b,a]$

so that $ab \in \mathcal{O}$. □

6.2. Margins

Reversal independence says that adding a ballot and its reverse should not change the result of the elections. In other words, a ballot and its reverse should always cancel each other out.

Definition:

Suppose w_1 and w_2 are strict weak orders with

(6.2.1) $\forall\ a,b \in A$: $a \succ_{w_1} b \Leftrightarrow b \succ_{w_2} a$.

Suppose $V^{\text{new}} := V^{\text{old}} + \{w_1\} + \{w_2\}$.

Then, an election method satisfies *reversal independence* if the following holds:

(6.2.2) $\mathcal{O}^{\text{new}} = \mathcal{O}^{\text{old}}$.

(6.2.3) $\mathcal{S}^{\text{new}} = \mathcal{S}^{\text{old}}$.

Claim:

If $\succ_{margin}$ is being used, then the Schulze method, as defined in section 2.2, satisfies reversal independence.

Proof:

The proof is trivial. When w_1 and w_2 are added, then $N^{\text{new}}[a,b] - N^{\text{new}}[b,a] = N^{\text{old}}[a,b] - N^{\text{old}}[b,a]$ for all $a,b \in A$. Therefore

(6.2.4) $\forall\ (a,b),(g,h) \in A \times A$:

$$((N^{\text{new}}[e,f] - N^{\text{new}}[f,e] > N^{\text{new}}[g,h] - N^{\text{new}}[h,g])$$

$$\Leftrightarrow (N^{\text{old}}[e,f] - N^{\text{old}}[f,e] > N^{\text{old}}[g,h] - N^{\text{old}}[h,g])).$$

Therefore

(6.2.5) $\forall\ (a,b),(g,h) \in A \times A$:

$$(N^{\text{new}}[e,f],N^{\text{new}}[f,e]) \succ_{margin} (N^{\text{new}}[g,h],N^{\text{new}}[h,g])$$

$$\Leftrightarrow (N^{\text{old}}[e,f],N^{\text{old}}[f,e]) \succ_{margin} (N^{\text{old}}[g,h],N^{\text{old}}[h,g]).$$

With (2.2.2) and (6.2.5), we get (6.2.2) and (6.2.3). □

7. Supermajority Requirements

When preferential ballots are being used in referendums, then sometimes alternatives have to fulfill some supermajority requirements to qualify. Typical supermajority requirements define some $M_1 \in \mathbb{N}$ or some $1 \leq M_2 \in \mathbb{R}$ and say that $N[a,b]$ must be strictly larger than max { $N[b,a]$, M_1 } or that $N[a,b]$ must be strictly larger than $M_2 \cdot N[b,a]$ to replace alternative $b \in A$ by alternative $a \in A$. Or they say that $N[a,b]$ must be strictly larger than $N[b,a]$ not only in the electorate as a whole, but also in a majority of its geographic parts or even in each of its geographic parts. It is also possible that in the same referendum the voters have to choose between alternatives that have to fulfill different supermajority requirements to qualify. In this section, we discuss a possible way to combine the Schulze method with supermajority requirements. Suppose $s \in A$ is the *status quo*.

These are the two tasks of supermajority requirements:

Task #1 (*protecting the status quo*):

Supermajority requirements protect the status quo from accidental majorities. They make it more difficult to replace the status quo s by alternative $a \in A \setminus \{s\}$. Therefore, an important property of all supermajority requirements is that, when s had won in the absence of these requirements, then it also wins in the presence of these requirements.

Task #2 (*preventing the status quo from cycling*):

Supermajority requirements prevent the status quo from cycling. Suppose $s(0)$ is the starting status quo. Suppose $s(k+1)$ is the new status quo when the method is applied to the same set of alternatives A, to the same set of ballots V, and to the status quo $s(k)$. Then we would expect that (for every possible set of alternatives A, for every possible set of ballots V, and for every possible starting status quo $s(0) \in A$) there is an $m < C$ such that $s(k) \equiv s(m)$ for all $k \geq m$.

We recommend the following method:

> The Schulze relation $\mathcal{O}$, as defined in section 2.2, is calculated.
>
> A *Tie-Breaking Ranking of the Links* (TBRL), a linear order $\succ_\sigma$ on $A \times A$, and a *Tie-Breaking Ranking of the Candidates* (TBRC), a linear order $\succ_\mu$ on A, are calculated as described in section 5.2 variant 1.
>
> The final Schulze relation $\mathcal{O}_{final}(\sigma)$, as defined in section 5.1, is calculated.
>
> Alternative $a \in A \setminus \{s\}$ is *attainable* if and only if $N[a,s] > N[s,a]$ and (a) there is no supermajority requirement to replace the status quo s by alternative a or (b) alternative a has the supermajority required to replace the status quo s by alternative a.
>
> Alternative $a \in A$ is *eligible* if and only if ($a \equiv s$) or ((a is attainable) and ($as \in \mathcal{O}$)).
>
> A winner is an alternative $a \in A$ with (1) alternative a is eligible and (2) $ab \in \mathcal{O}_{final}(\sigma)$ for every other eligible alternative b.

The condition “$as \in \mathcal{O}$” in the definition of eligibility implies that alternative a can win only if it had disqualified the status quo s in the absence of supermajority requirements. This guarantees that, if s had won in the absence of supermajority requirements, then s also wins in the presence of these supermajority requirements.

In the above suggestion, the status quo s can only be replaced by an alternative a with $as \in \mathcal{O}$. As $\mathcal{O}$ is transitive, it is guaranteed that the status quo cannot be changed in a cyclic manner.

8. Electoral College

There has been some debate about how to combine the Schulze method with the Electoral College for the elections of the President of the USA. In my opinion, the Electoral College serves two important purposes:

> Purpose #1: The Electoral College gives more power to the smaller states.
>
> > The Senate, where each state has the same voting power regardless of its population, is more powerful than the House of Representatives, where each state has a voting power in proportion of its population. This is true especially for decisions that are close to the executive. For example, the President needs the consent of the Senate for treaties and for the appointment of officers and judges. Because of this reason, it is more important that the President has a reliable support in the Senate than that he has a reliable support in the House of Representatives.
>
> Purpose #2: The Electoral College makes it possible to count the ballots on the state levels and then to add up the electoral votes.
>
> > The Electoral College makes it possible that, to guarantee that all voters are treated in an equal manner, it is only necessary to guarantee that all voters *in the same state* are treated in an equal manner. However, if the ballots were added up on the national level, it would be necessary to guarantee that *all voters all over the USA* are treated in an equal manner. In the latter case, many provisions (e.g. the rules to gain suffrage or to be excluded from suffrage, the ballot access rules, the rules for postal voting, the opening hours of the polling places) would have to be harmonized all over the USA, leading to a very powerful central election authority.
> >
> > This property is desirable especially for the elections to the National Conventions for the nominations of the presidential candidates. Here, the election rules and the set of candidates differ significantly from state to state.

To combine the Schulze method with the Electoral College without losing any of its purposes, we recommend that, for each pair of candidates a and b separately, we should determine, how many electoral votes $N_{electors}[a,b]$ candidate a would get and how many electoral votes $N_{electors}[b,a]$ candidate b would get when only these two candidates were running. We then apply the Schulze method to the matrix $N_{electors}$.

So we recommend the following method:

Stage 1:

Suppose $A_X \subseteq A$ is the set of candidates who are running in state X.

For $a,b \in A_X$: $N_X[a,b] \in \mathbb{N}_0$ is the number of voters in state X who strictly prefer candidate a to candidate b.

Stage 2:

Suppose $y \in \mathbb{R}$ with $y > 0$. Then “smaller_or_equal(y)” is the largest integer that is smaller than or equal to y. In other words: “smaller_or_equal(y)” is that integer $z \in \mathbb{N}_0$ with $z \leq y < (z + 1)$.

Suppose $y \in \mathbb{R}$ with $y > 0$. Then “strictly_smaller(y)” is the largest integer that is strictly smaller than y. In other words: “strictly_smaller(y)” is that integer $z \in \mathbb{N}_0$ with $z < y \leq (z + 1)$.

Suppose $E_X \in \mathbb{N}$ is the number of electors of state X.

Suppose:

(a) $F_X[a,b] := E_X$,

if $\{ a \in A_X$ and $b \notin A_X \}$ or $\{ a,b \in A_X$ and $N_X[a,b] > N_X[b,a] = 0 \}$.

(b) $F_X[a,b] := 0$,

if $\{ a \notin A_X$ and $b \in A_X \}$ or $\{ a,b \in A_X$ and $N_X[b,a] > N_X[a,b] = 0 \}$.

(c) $F_X[a,b] := E_X / 2$,

if $\{ a,b \notin A_X \}$ or $\{ a,b \in A_X$ and $N_X[a,b] = N_X[b,a] \}$.

(d) $F_X[a,b] := 0.01 \cdot \text{smaller\_or\_equal}\left(\dfrac{N_X[a,b] \cdot (1 + 100 \cdot E_X)}{N_X[a,b] + N_X[b,a]} \right)$,

if $a,b \in A_X$ and $N_X[a,b] > N_X[b,a] > 0$.

(e) $F_X[a,b] := 0.01 \cdot \text{strictly\_smaller}\left(\dfrac{N_X[a,b] \cdot (1 + 100 \cdot E_X)}{N_X[a,b] + N_X[b,a]} \right)$,

if $a,b \in A_X$ and $N_X[b,a] > N_X[a,b] > 0$.

$N_{electors}[a,b] := \sum_X F_X[a,b]$.

Stage 3:

The Schulze method, as defined in section 2.2, is applied to $N_{electors}$.

Suppose the Schulze method is used for presidential primaries. Suppose some candidate g withdraws and doesn’t take part in the remaining primaries. Then candidate g is not removed from the pairwise matrix. Rather he is treated as described at stage 2 (a) – (c). This regulation is necessary because removing a loser can still change the winner.

9. Comparison with other Methods

Table 9.2 compares the Schulze method with its main contenders. Extensive descriptions of the different methods can be found in publications by Fishburn (1977), Nurmi (1987), Kopfermann (1991), Levin and Nalebuff (1995), and Tideman (2006). As most of these methods only generate a set $\mathcal{S}$ of winners and don’t generate a binary relation $\mathcal{O}$, only that part of the different criteria is considered that refers to the set $\mathcal{S}$ of potential winners.

In terms of satisfied and violated criteria, that election method, that comes closest to the Schulze method, is Tideman’s ranked pairs method (Tideman, 1987). The only difference is that the ranked pairs method doesn’t choose from the MinMax set $\mathfrak{B}_D$.

The ranked pairs method works from the strongest to the weakest link. The link *xy* is locked if and only if it doesn’t create a directed cycle with already locked links. Otherwise, this link is locked in its opposite direction.

In example 1, the ranked pairs method locks *db*. Then it locks *cb*. Then it locks *ac*. Then it locks *ab*, since locking *ba* in its original direction would create a directed cycle with the already locked links *ac* and *cb*. Then it locks *cd*. Then it locks *ad*, since locking *da* in its original direction would create a directed cycle with the already locked links *ac* and *cd*.

The winner of the ranked pairs method is alternative $a \notin \mathfrak{B}_D = \{d\}$, because there is no locked link that ends in alternative *a*.

Although Tideman’s ranked pairs method is that election method that comes closest to the Schulze method in terms of satisfied and violated criteria, random simulations by Wright (2009) showed that that election method, that agrees the most frequently with the Schulze method, is the Simpson-Kramer method (table 9.1).

number of alternatives	A	B	C
3	100.0 %	100.0 %	100.0 %
4	99.7 %	98.5 %	98.2 %
5	99.2 %	96.0 %	95.3 %
6	99.1 %	93.0 %	92.3 %
7	98.9 %	90.0 %	89.1 %

Table 9.1: Simulations by Wright (2009)

A: Probability that the Schulze method conforms with the Simpson-Kramer method

B: Probability that the Schulze method conforms with the ranked pairs method

C: Probability that the ranked pairs method conforms with the Simpson-Kramer method

	resolvability	Pareto	reversal symmetry	monotonicity	independence of clones	Smith	Smith-IIA	Condorcet	Condorcet loser	majority for solid coalitions	majority	majority loser	participation	MinMax set	prudence	polynomial runtime
Baldwin	Y	Y	N	N	N	Y	N	Y	Y	Y	Y	Y	N	N	N	Y
Black	Y	Y	Y	Y	N	N	N	Y	Y	N	Y	Y	N	N	N	Y
Borda	Y	Y	Y	Y	N	N	N	N	Y	N	N	Y	Y	N	N	Y
Bucklin	Y	Y	N	Y	N	N	N	N	N	Y	Y	Y	N	N	N	Y
Copeland	N	Y	Y	Y	N	Y	Y	Y	Y	Y	Y	Y	N	N	N	Y
Dodgson	Y	Y	N	N	N	N	N	Y	N	N	Y	N	N	N	N	N
instant runoff	Y	Y	N	N	Y	N	N	N	Y	Y	Y	Y	N	N	N	Y
Kemeny-Young	Y	Y	Y	Y	N	Y	Y	Y	Y	Y	Y	Y	N	N	N	N
Nanson	Y	Y	Y	N	N	Y	N	Y	Y	Y	Y	Y	N	N	N	Y
plurality	Y	Y	N	Y	N	N	N	N	N	N	Y	N	Y	N	N	Y
ranked pairs	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	N	N	Y	Y
Schulze	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	Y
Simpson-Kramer	Y	Y	N	Y	N	N	N	Y	N	N	Y	N	N	N	Y	Y
Slater	N	Y	Y	Y	N	Y	Y	Y	Y	Y	Y	Y	N	N	N	N
Young	Y	Y	N	Y	N	N	N	Y	N	N	Y	N	N	N	N	N

Table 9.2: Comparison of Election Methods

"Y" = compliance
"N" = violation

10. Discussion

Suppose $\Lambda_D(a) := \max_D \{ (N[b,a],N[a,b]) \mid b \in A \setminus \{a\} \}$ is the Simpson-Kramer score of alternative $a \in A$. Then the Simpson-Kramer method is defined as follows:

(10.1) $a \in \mathcal{S}_{\text{SK}} :\Leftrightarrow \Lambda_D(a) \precsim_D \Lambda_D(b)$ for all $b \in A \setminus \{a\}$.

Over a long period of time, this method was the most popular election method among Condorcet activists, because this method minimizes the number of overruled voters. However, a very serious problem of this method is that it is not independent of clones, because it can happen that, when alternative $a \in A$ is replaced by a set of clones K as described in (4.6.1) – (4.6.3), then the alternatives of the set K disqualify each other in such a manner that for some alternative $b \in A \setminus \{a\}$:

(10.2) $\Lambda_D^{\text{old}}(a) \prec_D \Lambda_D^{\text{old}}(b)$ and $\Lambda_D^{\text{new}}(b) \prec_D \Lambda_D^{\text{new}}(g)\ \forall\, g \in K$.

To make the Simpson-Kramer method clone-proof, the concept of Simpson-Kramer scores has to be generalized from individual alternatives $a \in A$ to sets of alternatives $\emptyset \neq B \subsetneq A$:

(10.3) $\Gamma_D(B) := \max_D \{ (N[b,a],N[a,b]) \mid b \notin B, a \in B \}$.

We get

(10.4) $\forall\, a \in A$: $\Lambda_D(a) \approx_D \Gamma_D(\{a\})$.

The Γ_D scores are clone-proof because, when alternative $a \in A$ is replaced by a set of clones K, then we get for all $\emptyset \neq B \subsetneq A$:

(10.5a) $a \in B \Rightarrow \Gamma_D^{\text{new}}(\,(\,B \cup K\,) \setminus \{a\}\,) \approx_D \Gamma_D^{\text{old}}(B)$.

(10.5b) $a \notin B \Rightarrow \Gamma_D^{\text{new}}(B) \approx_D \Gamma_D^{\text{old}}(B)$.

Suppose $\beta_D := \min_D \{ \Gamma_D(B) \mid \emptyset \neq B \subsetneq A \}$ and $\mathfrak{B}_D := \cup \{ \emptyset \neq B \subsetneq A \mid \Gamma_D(B) \approx_D \beta_D \}$. Then when we want primarily that the used election method is clone-proof and secondarily that it minimizes the number of overruled voters, then the maximum, that we can ask for, is

(10.6) $\mathcal{S} \subseteq \mathfrak{B}_D$.

In this paper, we propose a new single-winner election method (*Schulze method*) that is clone-proof (section 4.6) and that always chooses from the MinMax set $\mathfrak{B}_D$ (section 4.8). The latter property is the most characteristic property of the Schulze method, since this is the first time that an election method with this property is proposed.

The Schulze method also satisfies many other criteria; some of them are also satisfied by the Simpson-Kramer method, like the Pareto criterion (section 4.3), resolvability (section 4.2), monotonicity (section 4.5), and prudence (section 4.9); some of them are violated by the Simpson-Kramer method, like the Smith criterion (section 4.7) and reversal symmetry (section 4.4). Because of this large number of satisfied criteria, we consider the

Schulze method to be a promising alternative to the Simpson-Kramer method for actual implementations, especially when manipulation through clones or weak alternatives is an issue.

Acknowledgments

I want to thank Lowell Bruce Anderson, Blake Cretney, Jobst Heitzig, Ross Hyman, Rob Lanphier, Rob LeGrand, Andrew Myers, Norman Petry, Nic Tideman, Kevin Venzke, Douglas R. Woodall, and Thomas Zavist for fruitful discussions.

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